第七节 斯托克斯公式 *环流量与旋度

一、斯托克斯公式

*二、空间曲线积分与路径无关的条件

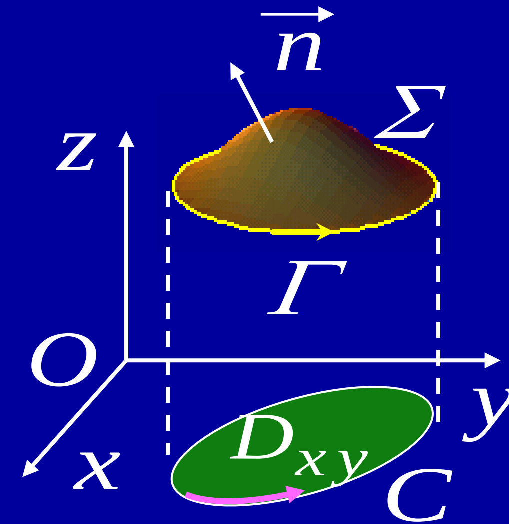
*三、环流量与旋度

一、 斯托克斯公式

定理1. 设光滑曲面Σ的边界Γ是分段光滑曲线, Σ 的侧与 Γ的正向符合右手法则, P, Q, R在包含Σ在内的一个空间域内具有连续一阶偏导数, 则有

$$\iint_{\Sigma}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}\,y\,\mathrm{d}\,z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}\,z\,\mathrm{d}\,x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}\,x\,\mathrm{d}\,y$$

$$=\oint_{\Gamma}P\,\mathrm{d}\,x+Q\,\mathrm{d}\,y+R\,\mathrm{d}\,z$$ (斯托克斯公式)



证: 情形1. Σ与平行 z 轴的直线只交于一点, 设其方程为

$$\Sigma: z=f(x,y),\quad (x,y)\in D_{xy}$$

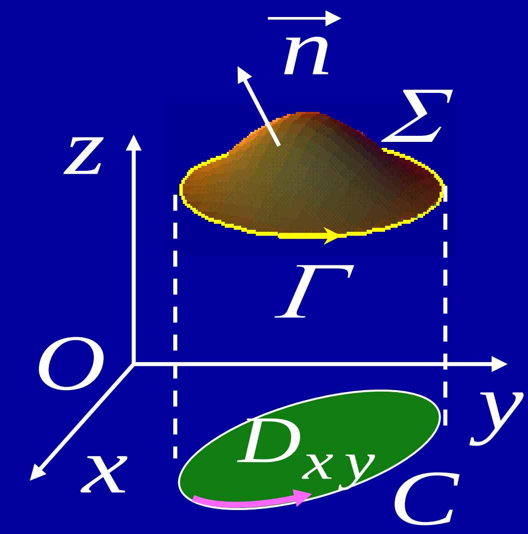
为确定起见, 不妨设Σ取上侧 (如图).

则 $\oint_{\Gamma} P\,\mathrm{d}x = \oint_{C} P(x,y,z(x,y))\,\mathrm{d}x$

$$= -\iint_{D_{xy}} \frac{\partial}{\partial y} P(x,y,z(x,y))\,\mathrm{d}x\,\mathrm{d}y \quad \text{(利用格林公式)}$$

$$= -\iint_{D_{xy}} \left[\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z}\frac{\partial z}{\partial y} \right] \mathrm{d}x\,\mathrm{d}y$$

$$= -\iint_{\Sigma} \left[\frac{\partial P}{\partial y} + \frac{\partial P}{\partial z} f_y \right] \cos\gamma\,\mathrm{d}S$$



$\because \quad \cos\gamma = \dfrac{1}{\sqrt{1+f_x^2+f_y^2}}, \qquad \cos\beta = \dfrac{-f_y}{\sqrt{1+f_x^2+f_y^2}},$

$\therefore \quad f_y = -\dfrac{\cos\beta}{\cos\gamma}$

因此 $\oint_{\Gamma} P\,\mathrm{d}x = -\iint_{\Sigma}\left[\dfrac{\partial P}{\partial y} - \dfrac{\partial P}{\partial z}\dfrac{\cos\beta}{\cos\gamma}\right]\cos\gamma\,\mathrm{d}S$

$$= \iint_{\Sigma}\left[\frac{\partial P}{\partial z}\cos\beta - \frac{\partial P}{\partial y}\cos\gamma\right]\mathrm{d}S$$

$$= \iint_{\Sigma}\frac{\partial P}{\partial z}\,\mathrm{d}z\,\mathrm{d}x - \frac{\partial P}{\partial y}\,\mathrm{d}x\,\mathrm{d}y$$

同理可证 $\oint_{\Gamma} Q\,\mathrm{d}y = \iint_{\Sigma}\dfrac{\partial Q}{\partial x}\,\mathrm{d}x\,\mathrm{d}y - \dfrac{\partial Q}{\partial z}\,\mathrm{d}y\,\mathrm{d}z$

$$\oint_{\Gamma} R\,\mathrm{d}x = \iint_{\Sigma}\frac{\partial R}{\partial y}\,\mathrm{d}y\,\mathrm{d}z - \frac{\partial R}{\partial x}\,\mathrm{d}z\,\mathrm{d}x$$

三式相加, 即得斯托克斯公式;

情形2 曲面Σ与平行 z 轴的直线交点多于一个, 则可通过作辅助线把 Σ分成与z 轴只交于一点的几部分, 在每一部分上应用斯托克斯公式, 然后相加, 由于沿辅助曲线方向相反的两个曲线积分相加刚好抵消, 所以对这类曲面斯托克斯公式仍成立. 证毕

注意: 如果Σ是 xOy 面上的一块平面区域, 则斯托克斯公式就是格林公式, 故格林公式是斯托克斯公式的特例.

$$\iint_{\Sigma}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$=\oint_{\Gamma}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

为便于记忆，斯托克斯公式还可写作:

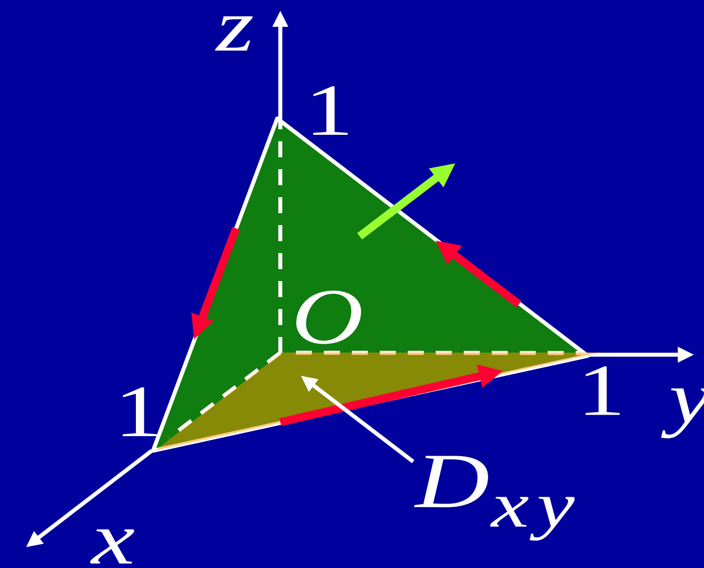
$$\iint_{\Sigma}\begin{vmatrix} \mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \oint_{\Gamma} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

或用第一类曲面积分表示:

$$\iint_{\Sigma}\begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}\mathrm{d}S = \oint_{\Gamma} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

例1. 利用斯托克斯公式计算积分 $\oint_{\Gamma} z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z$

其中 Γ 为平面 $x+y+z=1$ 被三坐标面所截三角形的整个边界, 方向如图所示.



解: 记三角形域为 Σ, 取上侧, 则

$$\oint_{\Gamma} z\,\mathrm{d}x + x\,\mathrm{d}y + y\,\mathrm{d}z$$

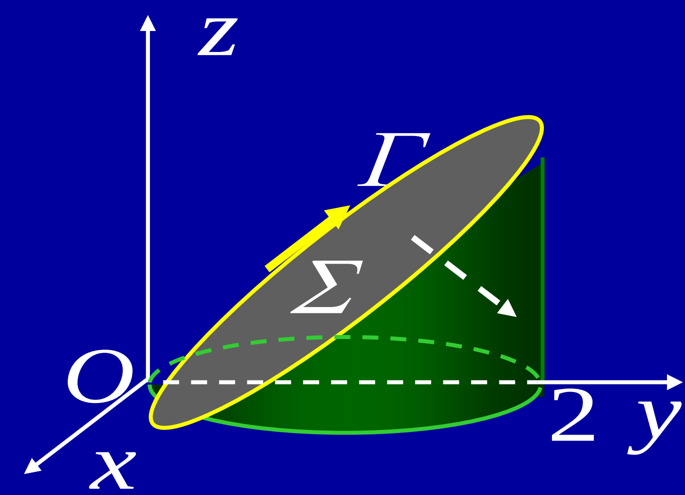
$$= \iint_{\Sigma} \begin{vmatrix} \mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ z & x & y \end{vmatrix}$$

$$= \iint_{\Sigma} \mathrm{d}y\,\mathrm{d}z + \mathrm{d}z\,\mathrm{d}x + \mathrm{d}x\,\mathrm{d}y = 3\iint_{D_{xy}} \mathrm{d}x\,\mathrm{d}y = \frac{3}{2}$$

利用对称性

例2. Γ 为柱面 $x^2+y^2=2y$ 与平面 $y=z$ 的交线, 从 z 轴正向看为顺时针, 计算 $I=\oint_\Gamma y^2\,\mathrm{d}x+xy\,\mathrm{d}y+xz\,\mathrm{d}z$.

解: 设 Σ 为平面 $z=y$ 上被 Γ 所围椭圆域, 且取下侧,



则其法线方向余弦

$$\cos\alpha=0\,,\ \cos\beta=\frac{1}{\sqrt{2}}\,,\ \cos\gamma=-\frac{1}{\sqrt{2}}$$

利用斯托克斯公式得

$$I=\iint_\Sigma\begin{vmatrix}\cos\alpha & \cos\beta & \cos\gamma\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & xy & xz\end{vmatrix}\mathrm{d}S=\frac{1}{\sqrt{2}}\iint_\Sigma(y-z)\,\mathrm{d}S=0$$

*二、空间曲线积分与路径无关的条件

定理2. 设 G 是空间一维单连通域, 函数 P,Q,R 在G内具有连续一阶偏导数, 则下列四个条件相互等价:

(1) 对G内任一分段光滑闭曲线Γ, 有

$$\oint_{\Gamma} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z = 0$$

(2) 对G内任一分段光滑曲线Γ, $\int_{\Gamma} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ 与路径无关

(3) 在G内存在某一函数 u, 使 $\mathrm{d}u = P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$

(4) 在G内处处有

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x},\quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y},\quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}$$

证: $(4)\Longrightarrow(1)$ 由斯托克斯公式可知结论成立;

$(1)\Longrightarrow(2)$ (自证)

$(2)\Longrightarrow(3)$ 设函数

$$u(x,y,z)=\int_{(x_0,y_0,z_0)}^{(x,y,z)}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

则

$$\frac{\partial u}{\partial x}=\lim_{\Delta x\to 0}\frac{u(x+\Delta x,y,z)-u(x,y,z)}{\Delta x}$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{(x,y,z)}^{(x+\Delta x,y,z)}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

$$=\lim_{\Delta x\to 0}\frac{1}{\Delta x}\int_{x}^{x+\Delta x}P\,\mathrm{d}x=\lim_{\Delta x\to 0}p(x+\Delta x,y,z)$$

$$=P(x,y,z)$$

同理可证 $\dfrac{\partial u}{\partial y}=Q(x,y,z),\quad \dfrac{\partial u}{\partial z}=R(x,y,z)$

故有 $\mathrm{d}u=P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$

$(3)\Longrightarrow(4)$ 若(3)成立, 则必有

$$\frac{\partial u}{\partial x}=P,\qquad \frac{\partial u}{\partial y}=Q,\qquad \frac{\partial u}{\partial z}=R$$

因P, Q, R 一阶偏导数连续, 故有

$$\frac{\partial P}{\partial y}=\frac{\partial^2 u}{\partial x\partial y}=\frac{\partial Q}{\partial x}$$

同理 $$\frac{\partial Q}{\partial z}=\frac{\partial R}{\partial y},\qquad \frac{\partial R}{\partial x}=\frac{\partial P}{\partial z}$$

证毕

例3. 验证曲线积分 $\int_{\Gamma}(y+z)\,\mathrm{d}x+(z+x)\,\mathrm{d}y+(x+y)\,\mathrm{d}z$

与路径无关, 并求函数

$$u(x,y,z)=\int_{(0,0,0)}^{(x,y,z)}(y+z)\mathrm{d}x+(z+x)\,\mathrm{d}y+(x+y)\,\mathrm{d}z$$

解: 令 $P=y+z\,,\ Q=z+x\,,\ R=x+y$

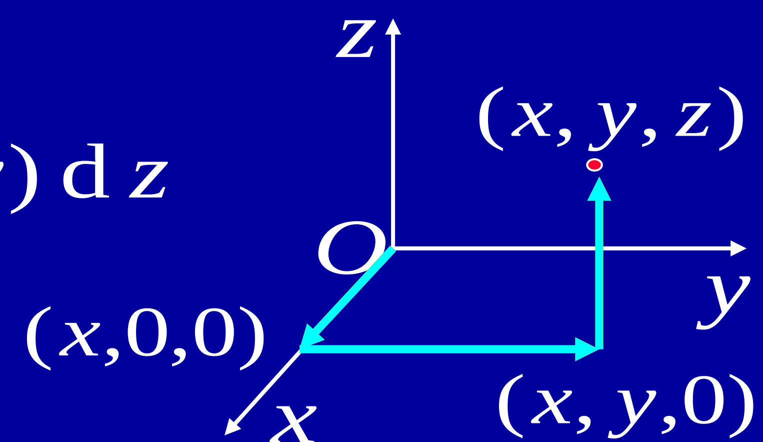
$$\because\quad \frac{\partial P}{\partial y}=1=\frac{\partial Q}{\partial x},\qquad \frac{\partial Q}{\partial z}=1=\frac{\partial R}{\partial y},\qquad \frac{\partial R}{\partial x}=1=\frac{\partial P}{\partial z}$$

$\therefore$ 积分与路径无关, 因此

$$u(x,y,z)=\int_0^x 0\,\mathrm{d}x+\int_0^y x\,\mathrm{d}y+\int_0^z (x+y)\,\mathrm{d}z$$

$$=xy+(x+y)z$$

$$=xy+yz+zx$$



*三、 环流量与旋度

斯托克斯公式

$$\iint_{\Sigma}\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\mathrm{d}y\,\mathrm{d}z+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\mathrm{d}z\,\mathrm{d}x+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\mathrm{d}x\,\mathrm{d}y$$

$$=\oint_{\Gamma}P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z$$

设曲面 Σ 的法向量为 $\vec{n}=(\cos\alpha,\cos\beta,\cos\gamma)$

曲线 Γ 的单位切向量为 $\vec{\tau}=(\cos\lambda,\cos\mu,\cos\nu)$

则斯托克斯公式可写为

$$\iint_{\Sigma}\left[\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}\right)\cos\alpha+\left(\frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}\right)\cos\beta+\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)\cos\gamma\right]\mathrm{d}S$$

$$=\oint_{\Gamma}(P\cos\lambda+Q\cos\mu+R\cos\nu)\,\mathrm{d}s$$

令 $\vec{A}=(P, Q, R)$, 引进一个向量

$$\left(\frac{\partial R}{\partial y}-\frac{\partial Q}{\partial z}, \frac{\partial P}{\partial z}-\frac{\partial R}{\partial x}, \frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)=\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$=\nabla\times\vec{A} \xlongequal{\text{记作}} \mathbf{rot}\vec{A}$$

rotation

于是得斯托克斯公式的向量形式：

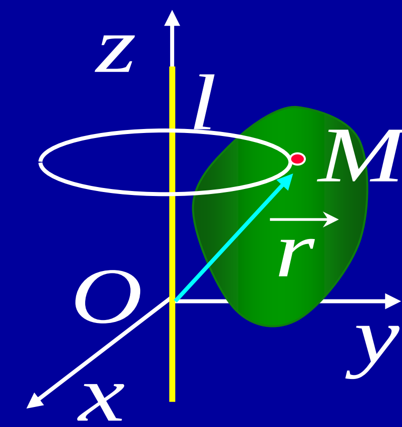
$$\iint_\Sigma \mathrm{rot}\,\vec{A}\cdot\vec{n}\,\mathrm{d}S=\oint_\Gamma \vec{A}\cdot\vec{\tau}\,\mathrm{d}s$$

或 $$\iint_\Sigma (\mathrm{rot}\,A)_n\,\mathrm{d}S=\oint_\Gamma A_\tau\,\mathrm{d}s \qquad ①$$

定义: $\oint_\Gamma P\,\mathrm{d}x+Q\,\mathrm{d}y+R\,\mathrm{d}z=\oint_\Gamma A_\tau\,\mathrm{d}s$ 称为向量场 $\vec{A}$ 沿有向闭曲线 Γ 的**环流量**. 向量 $\mathrm{rot}\,\vec{A}$ 称为向量场 $\vec{A}$ 的**旋度**.

旋度的力学意义:

设某刚体绕定轴 l 转动，角速度为 $\vec{\omega}$, M 为刚体上任一点，建立坐标系如图，则



$$\vec{\omega}=(0,0,\omega),\quad \vec{r}=(x,y,z)$$

点 M 的线速度为

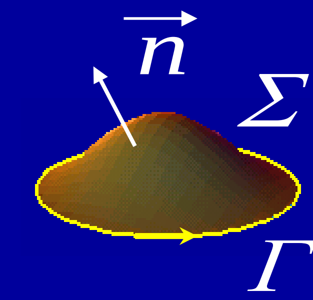
$$\vec{v}=\vec{\omega}\times\vec{r}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 0 & 0 & \omega\\ x & y & z\end{vmatrix}=(-\omega y,\omega x,0)$$

$$\operatorname{rot}\vec{v}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ -\omega y & \omega x & 0\end{vmatrix}=(0,0,2\omega)=2\vec{\omega}$$

(此即“旋度”一词的来源)

斯托克斯公式①的物理意义:

$$\iint_{\Sigma} (\operatorname{rot} A)_n \,\mathrm{d}S = \oint_{\Gamma} A_{\tau} \,\mathrm{d}s$$



向量场 $\vec{A}$ 产生的旋度场穿过 Σ 的通量

向量场 $\vec{A}$ 沿 Γ 的环流量

注意 Σ 与 Γ 的方向形成右手系!

例4. 求电场强度 $\vec{E} = \dfrac{q}{r^3}\vec{r}$ 的旋度 .

解:

$$\operatorname{rot} \vec{E} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{qx}{r^3} & \frac{qy}{r^3} & \frac{qz}{r^3} \end{vmatrix} = (0, 0, 0) \quad \text{(除原点外)}$$

这说明, 在除点电荷所在原点外, 整个电场无旋.

例5. 设 $\vec{A}=(2y\,,3x\,,z^2)$, $\Sigma: x^2+y^2+z^2=4$, $\vec{n}$ 为 Σ 的外法向量, 计算 $I=\oiint_{\Sigma}\mathrm{rot}\,\vec{A}\cdot\vec{n}\,\mathrm{d}S$.

解: $\mathrm{rot}\,\vec{A}=\nabla\times\vec{A}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ 2y & 3x & z^2\end{vmatrix}=(0\,,0\,,1)$

$$\vec{n}=(\cos\alpha\,,\cos\beta\,,\cos\gamma)$$

$$\therefore\quad I=\oiint_{\Sigma}\cos\gamma\,\mathrm{d}S=\oiint_{\Sigma_上}\mathrm{d}x\,\mathrm{d}y+\oiint_{\Sigma_下}\mathrm{d}x\,\mathrm{d}y$$

$$=\iint_{D_{xy}}\mathrm{d}x\,\mathrm{d}y-\iint_{D_{xy}}\mathrm{d}x\,\mathrm{d}y=0$$

内容小结

1. 斯托克斯公式

$$\oint_{\Gamma} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$$

$$= \iint_{\Sigma} \begin{vmatrix} \mathrm{d}y\,\mathrm{d}z & \mathrm{d}z\,\mathrm{d}x & \mathrm{d}x\,\mathrm{d}y \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

$$= \iint_{\Sigma} \begin{vmatrix} \cos\alpha & \cos\beta & \cos\gamma \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} \mathrm{d}S$$

也可写成:

$$\oint_{\Gamma} A_{\tau}\,\mathrm{d}s = \iint_{\Sigma} (\nabla \times \vec{A})_n\,\mathrm{d}S$$

其中

$\vec{A} = (P, Q, R)$

A_{τ} —— $\vec{A}$在 Γ 的切向量 $\vec{\tau}$ 上投影

$(\nabla \times \vec{A})_n$ —— $\vec{A}$的旋度 $\nabla \times \vec{A}$ 在 Σ 的法向量 $\vec{n}$ 上投影

2. 空间曲线积分与路径无关的充要条件

设 P, Q, R 在 Ω 内具有一阶连续偏导数, 则

$\int_{\Gamma} P\,\mathrm{d}x + Q\,\mathrm{d}y + R\,\mathrm{d}z$ 在 Ω 内与路径无关

$\Longleftrightarrow$ 在 Ω 内处处有

$$\frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z}, \quad \frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

$\Longleftrightarrow$ 在 Ω 内处处有

$$\mathrm{rot}(P, Q, R) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \vec{0}$$

3. 场论中的三个度

设 $u = u(x, y, z)$， $\vec{A} = (P, Q, R)$， $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z})$， 则

梯度: $$\operatorname{grad} u = \left(\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial u}{\partial z}\right) = \nabla u$$

散度: $$\operatorname{div} \vec{A} = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \nabla \cdot \vec{A}$$

旋度: $$\operatorname{rot} \vec{A} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \nabla \times \vec{A}$$

思考与练习

设 $r=\sqrt{x^2+y^2+z^2}$，则

$$\operatorname{div}(\operatorname{grad} r)=\underline{\ {}^{2}\!/_{r}\ }\,;\quad \operatorname{rot}(\operatorname{grad} r)=\underline{\ \vec{0}\ }\,.$$

提示: $\operatorname{grad} r=\left(\frac{x}{r},\frac{y}{r},\frac{z}{r}\right)$

$$\frac{\partial}{\partial x}\left(\frac{x}{r}\right)=\frac{r-x\cdot {}^{x}\!/_{r}}{r^2}=\frac{r^2-x^2}{r^3},\qquad \frac{\partial}{\partial y}\left(\frac{y}{r}\right)=\frac{r^2-y^2}{r^3}$$

$$\frac{\partial}{\partial z}\left(\frac{z}{r}\right)=\frac{r^2-z^2}{r^3}$$

三式相加即得 $\operatorname{div}(\operatorname{grad} r)$

$$\operatorname{rot}(\operatorname{grad} r)=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ \frac{x}{r} & \frac{y}{r} & \frac{z}{r}\end{vmatrix}=(0,0,0)$$

作业

P243　*2 (1),(4) ;　*3(1),(3) ;　*4(1);

*5 (2) ;　*7

补充题: 证明　$$\nabla\cdot(\nabla\times\vec{A})=0$$

(即 $\mathrm{div}\,(\mathbf{rot}\,\vec{A})=0$)