第二节

二重积分的计算法

一、利用直角坐标计算二重积分

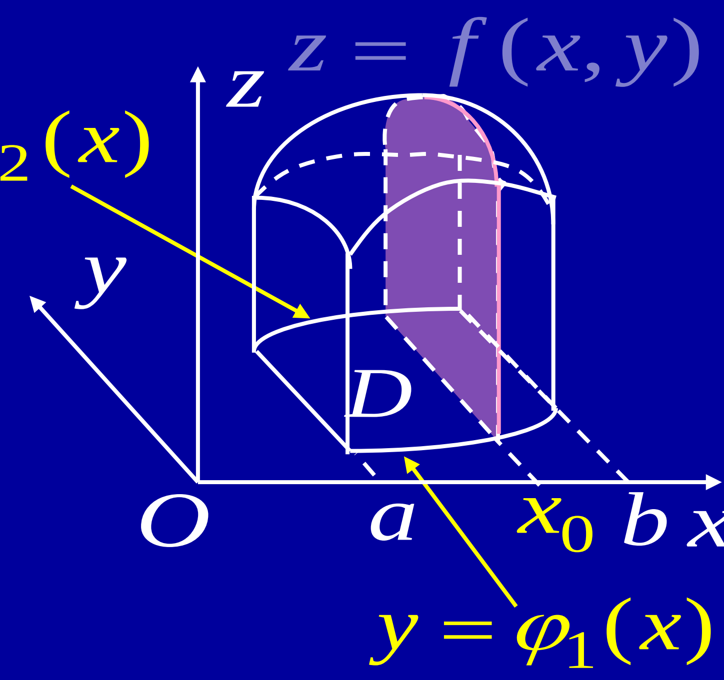
二、利用极坐标计算二重积分

*三、二重积分的换元法

一、利用直角坐标计算二重积分

设曲顶柱体的底为

$$D=\left\{(x,y)\middle|\begin{array}{c}a\le x\le b\\ \varphi_1(x)\le y\le\varphi_2(x)\end{array}\right\}$$



任取$x_0\in[a,b]$,平面$x=x_0$ 截柱体的

截面积为 $A(x_0)=\int_{\varphi_1(x_0)}^{\varphi_2(x_0)}f(x_0,y)\,\mathrm{d}y$

故曲顶柱体体积为

$$V=\iint_D f(x,y)\,\mathrm{d}\sigma=\int_a^b A(x)\,\mathrm{d}x$$

$$=\int_a^b\left[\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)\,\mathrm{d}y\right]\mathrm{d}x\overset{\text{记作}}{=\!=}\int_a^b\mathrm{d}x\int_{\varphi_1(x)}^{\varphi_2(x)}f(x,y)\,\mathrm{d}y$$

同样, 曲顶柱的底为

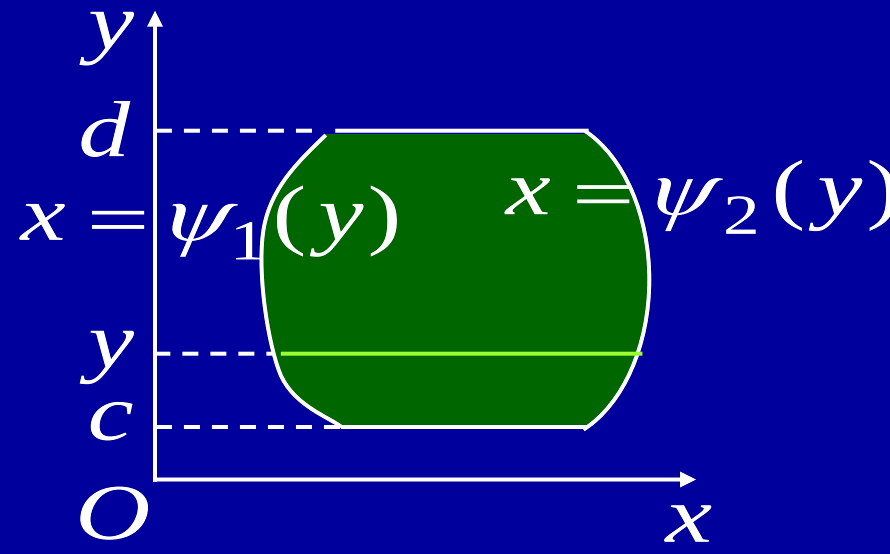
$$D=\{(x,y)\ |\psi_1(y)\le x\le\psi_2(y),\ c\le y\le d\ \}$$

则其体积可按如下两次积分计算

$$V=\iint_D f(x,y)\,\mathrm{d}\sigma$$

$$=\int_c^d\left[\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x\ \right]\mathrm{d}\,y$$

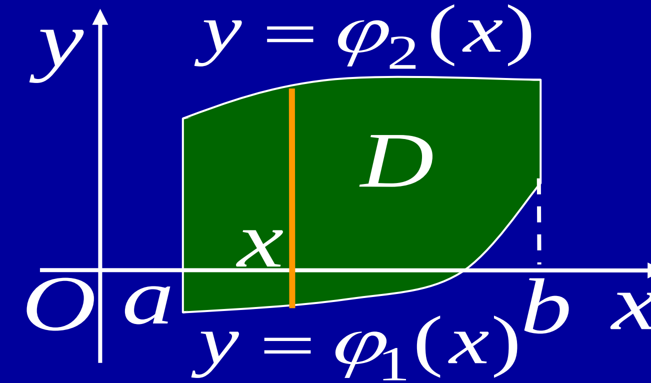
$$\overset{\text{记作}}{=}\int_c^d \mathrm{d}\,y\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$$



由曲顶柱体体积的计算可知，当被积函数 $f(x,y)\ge 0$

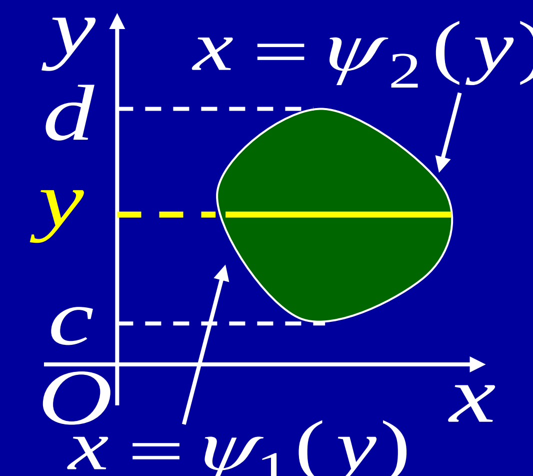
且在D上连续时，若D为 X - 型区域

$$D:\begin{cases} a\le x\le b \\ \varphi_1(x)\le y\le \varphi_2(x)\end{cases}$$



则 $\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y=\int_a^b \mathrm{d}\,x\int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y$

若D为Y - 型区域 $D:\begin{cases}\psi_1(y)\le x\le \psi_2(y)\\ c\le y\le d\end{cases}$



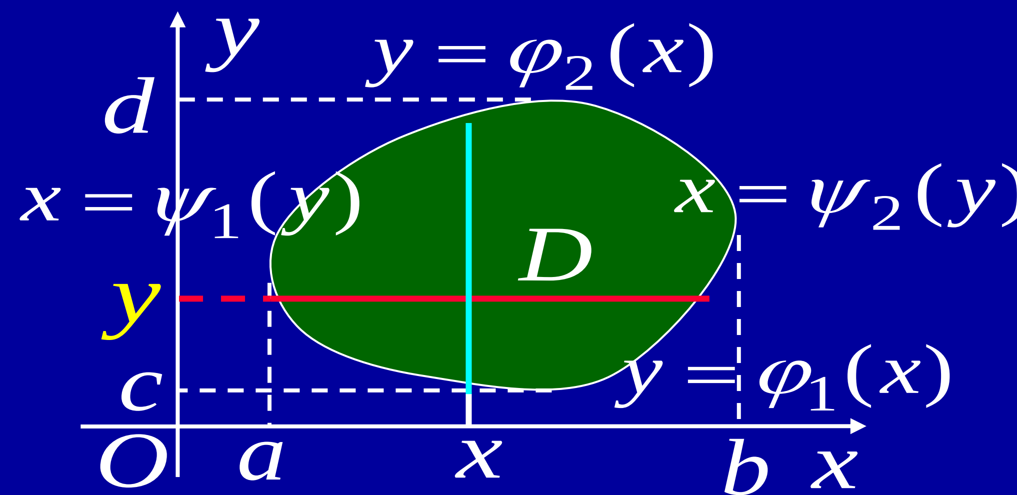
则 $\iint_D f(x,y)\,\mathrm{d}x\mathrm{d}y=\int_c^d \mathrm{d}\,y\int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$

说明: (1) 若积分区域既是 X - 型区域又是 Y - 型区域，

则有 $\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$

$$= \int_a^b \mathrm{d}\,x \int_{\varphi_1(x)}^{\varphi_2(x)} f(x,y)\,\mathrm{d}y$$

$$= \int_c^d \mathrm{d}\,y \int_{\psi_1(y)}^{\psi_2(y)} f(x,y)\,\mathrm{d}x$$

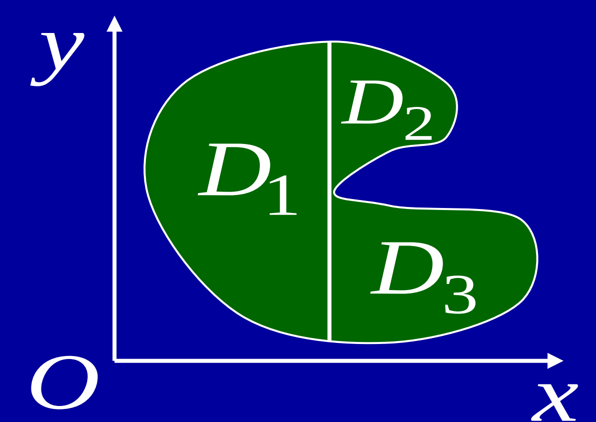


为计算方便,可**选择积分序**, 必要时还可以**交换积分序**.

(2) 若积分域较复杂,可将它分成若干

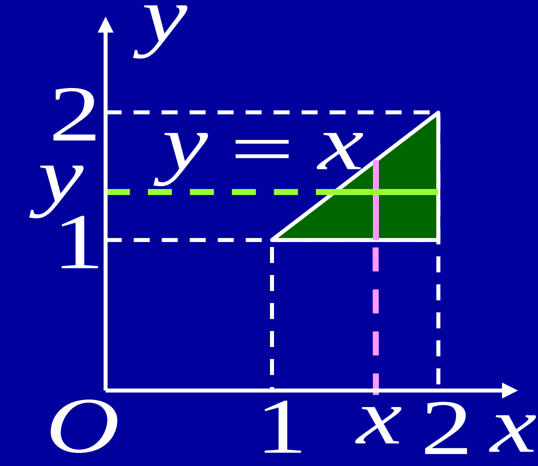
X - 型域或 Y - 型域，则

$$\iint_D = \iint_{D_1} + \iint_{D_2} + \iint_{D_3}$$



例1. 计算 $I=\iint_D xy\,\mathrm{d}\sigma$, 其中 D 是直线 $y=1, x=2$, 及 $y=x$ 所围的闭区域.

解法1. 将 D 看作 X - 型区域, 则 $D:\begin{cases}1\le x\le 2\\ 1\le y\le x\end{cases}$



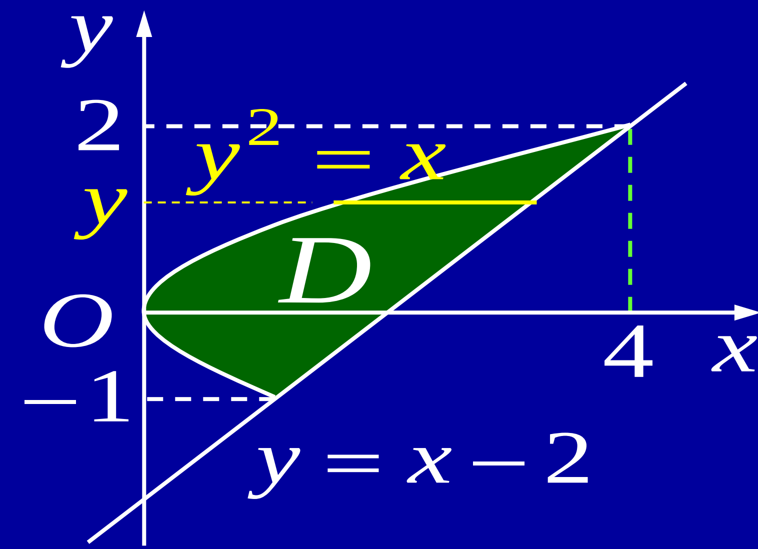
$$I=\int_1^2 \mathrm{d}\,x\int_1^x xy\,\mathrm{d}\,y=\int_1^2\left[\tfrac{1}{2}xy^2\right]_1^x\mathrm{d}\,x$$

$$=\int_1^2\left[\tfrac{1}{2}x^3-\tfrac{1}{2}x\right]\mathrm{d}x=\frac{9}{8}$$

解法2. 将 D 看作 Y - 型区域, 则 $D:\begin{cases}y\le x\le 2\\ 1\le y\le 2\end{cases}$

$$I=\int_1^2 \mathrm{d}\,y\int_y^2 xy\,\mathrm{d}\,x=\int_1^2\left[\tfrac{1}{2}x^2y\right]_y^2\mathrm{d}y=\int_1^2\left[2y-\tfrac{1}{2}y^3\right]\mathrm{d}y=\frac{9}{8}$$

例3. 计算 $\iint_D xy\,\mathrm{d}\sigma$, 其中$D$是抛物线 $y^2 = x$ 及直线 $y = x - 2$ 所围成的闭区域.



解: 为计算简便, 先对 x 后对 y 积分,

则 $D:\begin{cases} y^2 \le x \le y+2 \\ -1 \le y \le 2 \end{cases}$

$$\therefore \iint_D xy\,\mathrm{d}\sigma = \int_{-1}^{2} \mathrm{d}y \int_{y^2}^{y+2} xy\,\mathrm{d}x$$

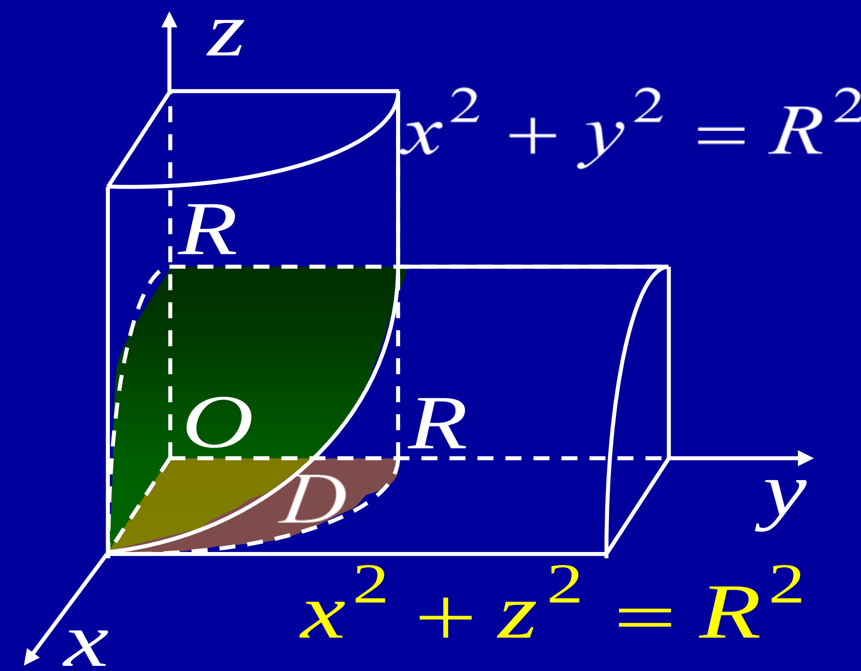
$$= \int_{-1}^{2} \left[\frac{1}{2}x^2 y\right]_{y^2}^{y+2} \mathrm{d}y = \frac{1}{2}\int_{-1}^{2} [y(y+2)^2 - y^5]\,\mathrm{d}y$$

$$= \frac{1}{2}\left[\frac{y^4}{4} + \frac{4}{3}y^3 + 2y^2 - \frac{1}{6}y^6\right]_{-1}^{2} = \frac{45}{8}$$

例4. 求两个底圆半径为R的直交圆柱面所围的体积.

解: 设两个直圆柱方程为

$$x^2+y^2=R^2,\quad x^2+z^2=R^2$$



利用对称性, 考虑第一卦限部分,

其曲顶柱体的顶为 $z=\sqrt{R^2-x^2}$

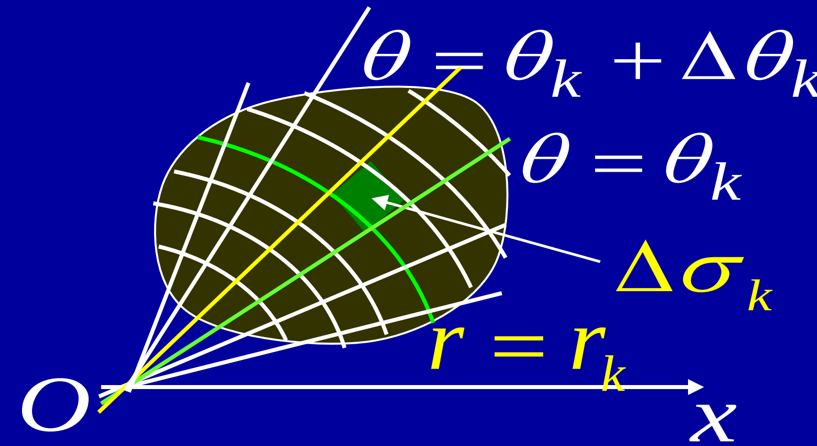
$$(x,y)\in D:\begin{cases}0\le x\le R\\ 0\le y\le\sqrt{R^2-x^2}\end{cases}$$

则所求体积为

$$V=8\iint_D\sqrt{R^2-x^2}\,\mathrm{d}x\,\mathrm{d}y=8\int_0^R\sqrt{R^2-x^2}\,\mathrm{d}x\int_0^{\sqrt{R^2-x^2}}\mathrm{d}y$$

$$=8\int_0^R(R^2-x^2)\,\mathrm{d}x=\frac{16}{3}R^3$$

二、利用极坐标计算二重积分



在极坐标系下, 用同心圆 $r=$常数

及射线 θ =常数, 分划区域D 为

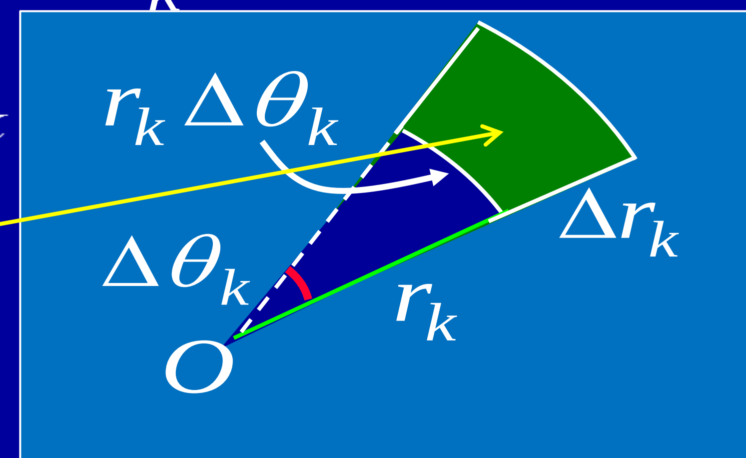
$$\Delta\sigma_k \quad (k=1,2,\cdots,n)$$

则除包含边界点的小区域外,小区域的面积

$$\Delta\sigma_k=\tfrac{1}{2}(r_k+\Delta r_k)^2\cdot\Delta\theta_k-\tfrac{1}{2}r_k{}^2\cdot\Delta\theta_k$$

$$=\underline{\tfrac{1}{2}[r_k+(r_k+\Delta r_k)]}\Delta r_k\cdot\Delta\theta_k$$

$$=\overline{r_k}\Delta r_k\cdot\Delta\theta_k$$

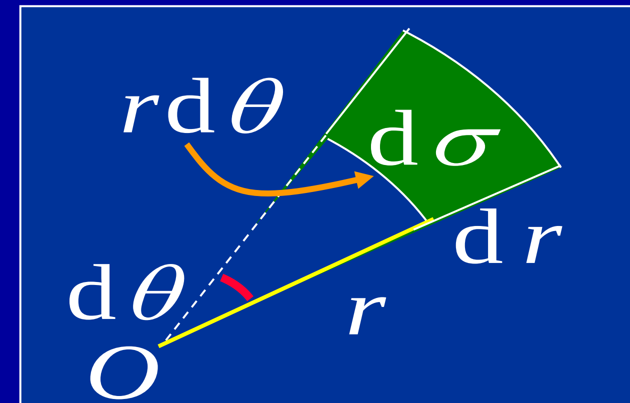


在 $\Delta\sigma_k$ 内取点 $(\overline{r_k},\overline{\theta_k})$, 对应有

$$\xi_k=\overline{r_k}\cos\overline{\theta_k},\ \eta_k=\overline{r_k}\sin\overline{\theta_k}$$

$$\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\xi_k,\eta_k)\Delta\sigma_k$$

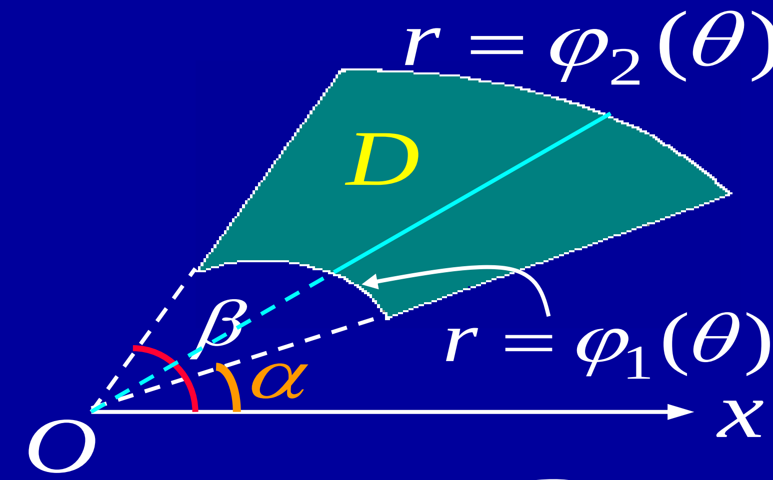
$$=\lim_{\lambda\to 0}\sum_{k=1}^{n} f(\overline{r_k}\cos\overline{\theta_k},\ \overline{r_k}\sin\overline{\theta_k})\overline{r_k}\Delta r_k\Delta\theta_k$$



即 $$\iint_D f(x,y)\,\mathrm{d}\sigma=\iint_D f(r\cos\theta,r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

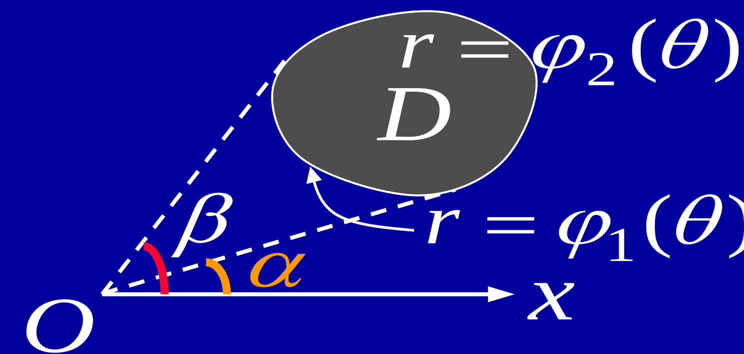
$$\iint_D f(x,y)\,\mathrm{d}\sigma=\iint_D f(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

设 $D:\begin{cases}\alpha\le\theta\le\beta\\ \varphi_1(\theta)\le r\le\varphi_2(\theta)\end{cases}$, 则

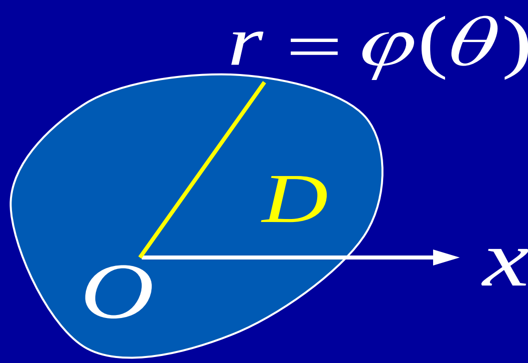


$$\iint_D f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=\int_\alpha^\beta \mathrm{d}\theta\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta, r\sin\theta)r\,\mathrm{d}r$$



特别, 对 $D:\begin{cases}0\le\theta\le 2\pi\\ 0\le r\le\varphi(\theta)\end{cases}$

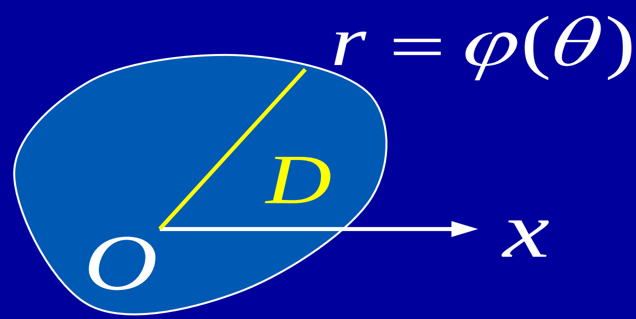


$$\iint_D f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=\int_0^{2\pi}\mathrm{d}\theta\int_0^{\varphi(\theta)} f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r$$

此时若 $f\equiv 1$ 则可求得D 的面积

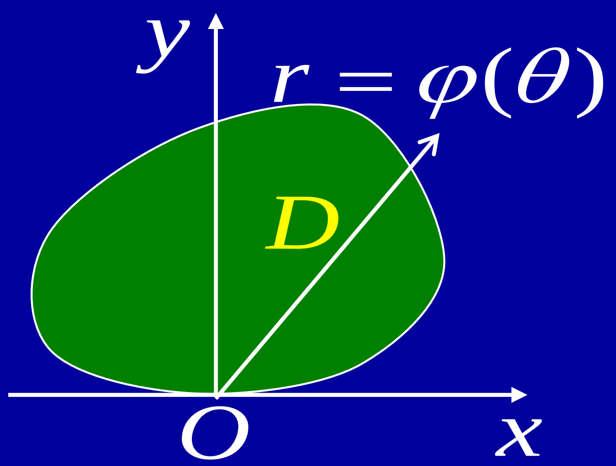
$$\sigma=\iint_D \mathrm{d}\sigma=\frac{1}{2}\int_0^{2\pi}\varphi^2(\theta)\,\mathrm{d}\theta$$



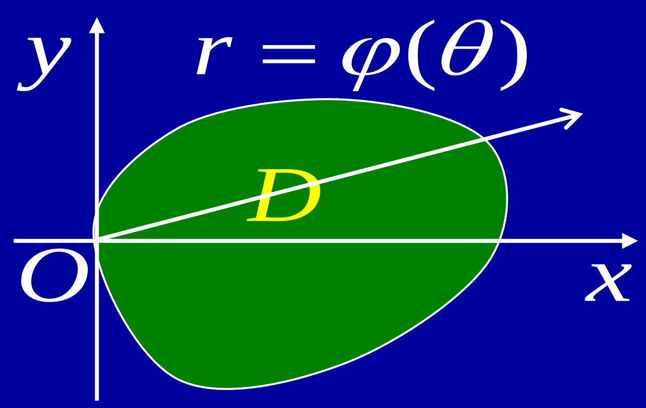
思考: 下列各图中域 D 分别与 x,y 轴相切于原点，

试问 θ 的变化范围是什么?

(1)

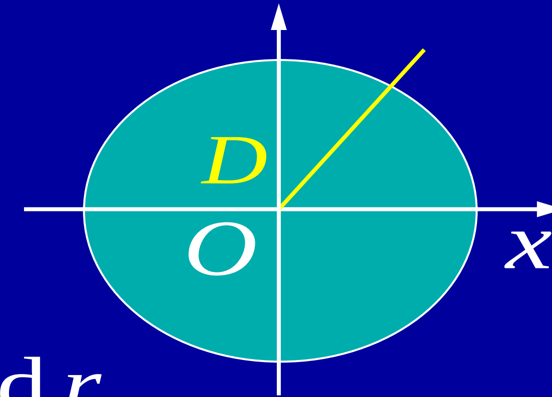


(2)



答: (1) $0\le\theta\le\pi$;　　(2) $-\dfrac{\pi}{2}\le\theta\le\dfrac{\pi}{2}$

例6. 计算 $\iint_D e^{-x^2-y^2}\,dx\,dy$，其中 $D: x^2+y^2\le a^2$.



解: 在极坐标系下 $D:\begin{cases}0\le r\le a\\0\le\theta\le 2\pi\end{cases}$，故

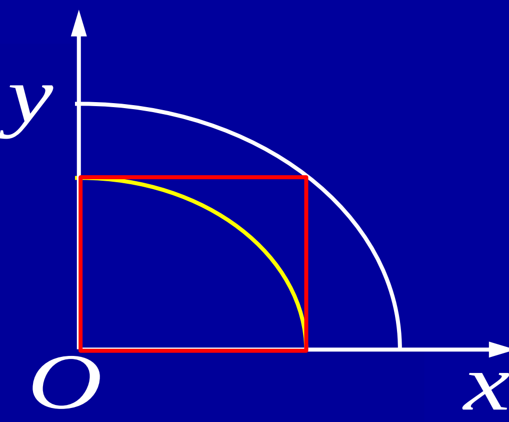
原式 $=\iint_D e^{-r^2}r\,dr\,d\theta=\int_0^{2\pi}d\theta\int_0^a re^{-r^2}\,dr$

$$=2\pi\left[\frac{-1}{2}e^{-r^2}\right]_0^a=\pi(1-e^{-a^2})$$

由于e^{-x^2}的原函数不是初等函数，故本题无法用直角坐标计算.

注： 利用上题可得一个在概率论与数理统计及工程上非常有用的反常积分公式

$$\int_0^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}\,x = \frac{\sqrt{\pi}}{2} \qquad ①$$



设 $D_1 = \{(x, y) \mid x^2 + y^2 \le R^2, x \ge 0, y \ge 0\}$

$S = \{(x, y) \mid 0 \le x \le R, 0 \le y \le R\}$

$D_2 = \{(x, y) \mid x^2 + y^2 \le 2R^2, x \ge 0, y \ge 0\}$

则 $$\iint_{D_1} \mathrm{e}^{-x^2-y^2}\,dxdy \le \iint_S \mathrm{e}^{-x^2-y^2}\,dxdy \le \iint_{D_2} \mathrm{e}^{-x^2-y^2}\,dxdy$$

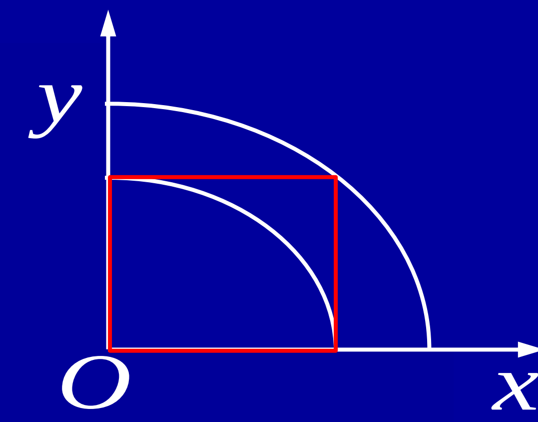
即 $$\frac{\pi}{4}(1-\mathrm{e}^{-R^2}) \le \iint_S \mathrm{e}^{-x^2-y^2}\,dxdy \le \frac{\pi}{4}(1-\mathrm{e}^{-2R^2})$$

$$\frac{\pi}{4}(1-e^{-R^2}) \le \iint_S e^{-x^2-y^2}\,dxdy \le \frac{\pi}{4}(1-e^{-2R^2})$$

$$\iint_S e^{-x^2-y^2}\,dxdy$$

$$= \int_0^R e^{-x^2}\,dx \int_0^R e^{-y^2}\,dy$$

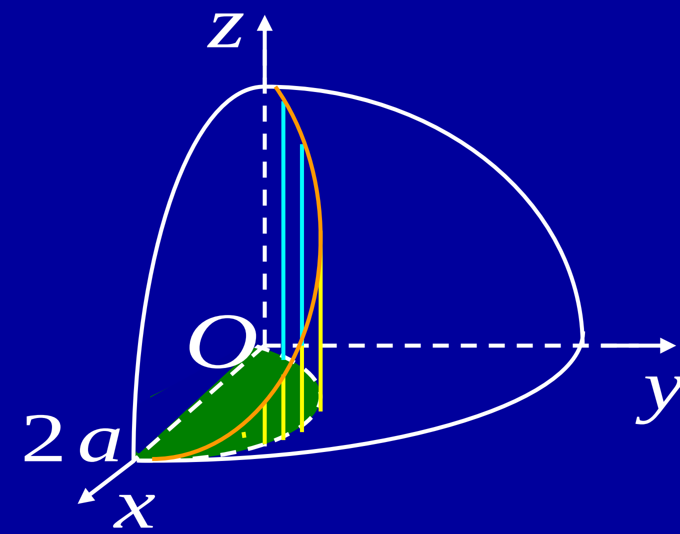
$$= \left(\int_0^R e^{-x^2}\,dx\right)^2$$



$$\frac{\pi}{4} \longleftarrow \frac{\pi}{4}(1-e^{-R^2}) \le \left(\int_0^R e^{-x^2}\,dx\right)^2 \le \frac{\pi}{4}(1-e^{-2R^2}) \longrightarrow \frac{\pi}{4}$$

因此 $\displaystyle\int_0^{+\infty} e^{-x^2}\,\mathrm{d}\,x = \frac{\sqrt{\pi}}{2}$

例7. 求球体 $x^2+y^2+z^2\le 4a^2$ 被圆柱面 $x^2+y^2=2ax$ $(a>0)$ 所截得的(含在柱面内的)立体的体积.



解: 设 $D: 0\le\theta\le\frac{\pi}{2},\ 0\le r\le 2a\cos\theta$

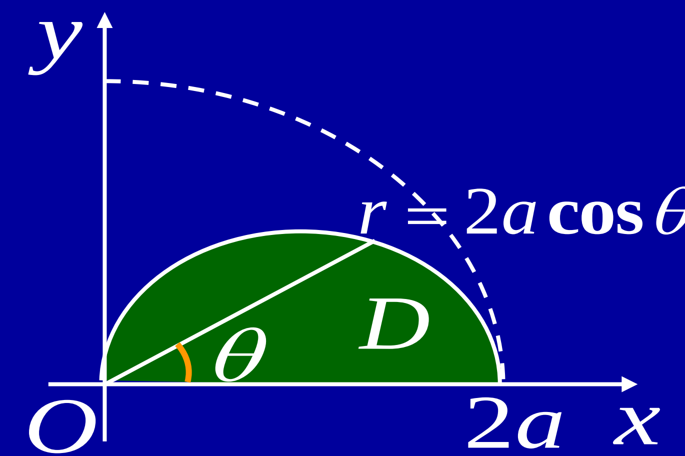
由对称性可知

$$V=4\iint_D\sqrt{4a^2-r^2}\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=4\int_0^{\pi/2}\mathrm{d}\theta\int_0^{2a\cos\theta}\sqrt{4a^2-r^2}\,r\,\mathrm{d}r$$

$$=\frac{32}{3}a^3\int_0^{\pi/2}(1-\sin^3\theta)\,\mathrm{d}\theta$$

$$=\frac{32}{3}a^3\left(\frac{\pi}{2}-\frac{2}{3}\right)$$



10.2 作业

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1 (2), (4); 2 (3), (4); 5; 6 (2), (4);

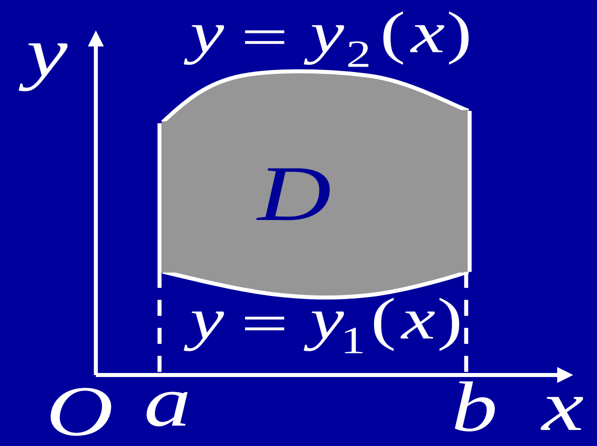
11(2), (4); 13 (3), (4); 14 (2), (3);

15 (1), (4); 16 18

内容小结

(1) 二重积分化为二次积分的方法

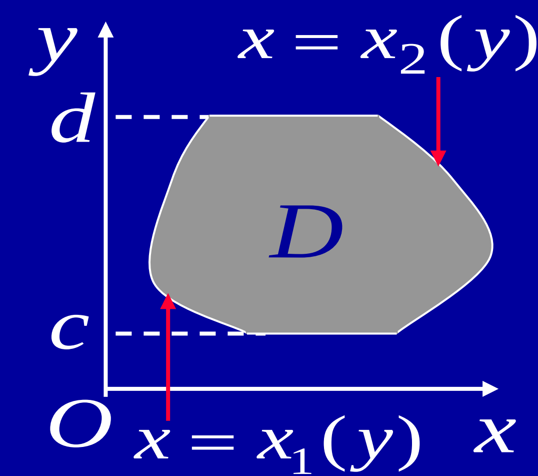
直角坐标系情形：



- 若积分区域为

$$D = \{(x, y) \mid a \le x \le b,\ y_1(x) \le y \le y_2(x)\}$$

则 $$\iint_D f(x, y)\,\mathrm{d}\sigma = \int_a^b \mathrm{d}x \int_{y_1(x)}^{y_2(x)} f(x, y)\,\mathrm{d}y$$



- 若积分区域为

$$D = \{(x, y) \mid c \le y \le d,\ x_1(y) \le x \le x_2(y)\}$$

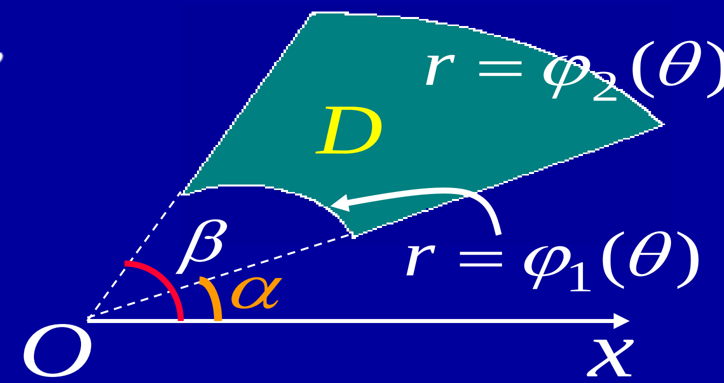
则 $$\iint_D f(x, y)\,\mathrm{d}\sigma = \int_c^d \mathrm{d}y \int_{x_1(y)}^{x_2(y)} f(x, y)\,\mathrm{d}x$$

极坐标系情形: 若积分区域为

$$D=\{(r,\theta)\,|\,\alpha\le\theta\le\beta\,,\ \varphi_1(\theta)\le r\le\varphi_2(\theta)\,\}$$

则 $$\iint_D f(x,y)\,\mathrm{d}\sigma=\iint_D f(r\cos\theta,r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$$

$$=\int_\alpha^\beta\mathrm{d}\theta\int_{\varphi_1(\theta)}^{\varphi_2(\theta)} f(r\cos\theta,r\sin\theta)\,r\,\mathrm{d}r$$



(2) 一般换元公式

在变换 $\begin{cases}x=x(u,v)\\ y=y(u,v)\end{cases}$ 下

$$(x,y)\in D\longleftrightarrow(u,v)\in D',\text{且}\quad J=\frac{\partial(x,y)}{\partial(u,v)}\ne 0$$

则 $$\iint_D f(x,y)\,\mathrm{d}\sigma=\iint_{D'} f[x(u,v),y(u,v)]\,|\,J\,|\,\mathrm{d}u\,\mathrm{d}v$$

(3) 计算步骤及注意事项

- 画出积分域
- 选择坐标系
  - 域边界应尽量多为坐标线
  - 被积函数关于坐标变量易分离
- 确定积分序
  - 积分域分块要少
  - 累次积分好算为妙
- 写出积分限
  - 图示法
  - 不等式

  (先积一条线, 后扫积分域)
- 计算要简便
  - 充分利用对称性
  - 应用换元公式