

第二节

函数的求导法则

- 一、四则运算求导法则
- 二、反函数的求导法则
- 三、复合函数求导法则
- 四、初等函数的求导问题



解决求导问题的思路:

$$f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \quad (\text{构造性定义})$$



本节内容



求导法则



其他基本初等函数
求导公式

$$\left\{ \begin{array}{l} (C)' = 0 \\ (\sin x)' = \cos x \\ (\ln x)' = \frac{1}{x} \end{array} \right\} \begin{array}{l} \text{证明中利用了} \\ \text{两个重要极限} \end{array}$$

初等函数求导问题



一、四则运算求导法则

定理1. 函数 $u = u(x)$ 及 $v = v(x)$ 都在点 x 可导

$\implies u(x)$ 及 $v(x)$ 的和、差、积、商 (除分母为 0 的点外)
都在点 x 可导, 且

$$(1) [u(x) \pm v(x)]' = u'(x) \pm v'(x)$$

$$(2) [u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$$

$$(3) \left[\frac{u(x)}{v(x)} \right]' = \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)} \quad (v(x) \neq 0)$$

下面分三部分加以证明, 并同时给出相应的推论和例题.



$$(1) \quad (u \pm v)' = u' \pm v'$$

证: 设 $f(x) = u(x) \pm v(x)$, 则

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h \rightarrow 0} \frac{[u(x+h) \pm v(x+h)] - [u(x) \pm v(x)]}{h} \\ &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} \pm \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} \\ &= u'(x) \pm v'(x) \quad \text{故结论成立.} \end{aligned}$$

此法则可推广到任意有限项的情形.

例如, $(u + v - w)' = u' + v' - w'$



$$(2) \quad (uv)' = u'v + uv'$$

证: 设 $f(x) = u(x)v(x)$, 则有

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} v(x+h) + u(x) \frac{v(x+h) - v(x)}{h} \right] \\ &= u'(x)v(x) + u(x)v'(x) \quad \text{故结论成立.} \end{aligned}$$

推论: 1) $(Cu)' = Cu'$ (C 为常数)

$$2) \quad (uvw)' = u'vw + uv'w + uvw'$$

$$3) \quad (\log_a x)' = \left(\frac{\ln x}{\ln a} \right)' = \frac{1}{x \ln a}$$



例1. $y = \sqrt{x}(x^3 - 4\cos x - \sin 1)$, 求 y' 及 $y'|_{x=1}$.

解:
$$y' = (\sqrt{x})'(x^3 - 4\cos x - \sin 1) + \sqrt{x}(x^3 - 4\cos x - \sin 1)'$$
$$= \frac{1}{2\sqrt{x}}(x^3 - 4\cos x - \sin 1) + \sqrt{x}(3x^2 + 4\sin x)$$

$$y'|_{x=1} = \frac{1}{2}(1 - 4\cos 1 - \sin 1) + (3 + 4\sin 1)$$
$$= \frac{7}{2} - 2\cos 1 + \frac{7}{2}\sin 1$$



$$(3) \quad \left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

证: 设 $f(x) = \frac{u(x)}{v(x)}$, 则有

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{\frac{u(x+h)}{v(x+h)} - \frac{u(x)}{v(x)}}{h} \\ &= \lim_{h \rightarrow 0} \left[\frac{\frac{u(x+h) - u(x)}{h} \cdot v(x) - u(x) \cdot \frac{v(x+h) - v(x)}{h}}{v(x+h)v(x)} \right] \\ &= \frac{u'(x)v(x) - u(x)v'(x)}{v^2(x)} \quad \text{故结论成立.} \end{aligned}$$

推论: $\left(\frac{C}{v}\right)' = \frac{-Cv'}{v^2} \quad (C \text{ 为常数})$



例2. 求证 $(\tan x)' = \sec^2 x$, $(\csc x)' = -\csc x \cot x$.

证:
$$(\tan x)' = \left(\frac{\sin x}{\cos x} \right)' = \frac{(\sin x)' \cos x - \sin x (\cos x)'}{\cos^2 x}$$
$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \sec^2 x$$

$$(\csc x)' = \left(\frac{1}{\sin x} \right)' = \frac{-(\sin x)'}{\sin^2 x} = \frac{-\cos x}{\sin^2 x}$$
$$= -\csc x \cot x$$

类似可证: $(\cot x)' = -\csc^2 x$, $(\sec x)' = \sec x \tan x$.



二、反函数的求导法则

定理2. 设 $y = f^{-1}(x)$ 为 $x = f(y)$ 的反函数, $x = f(y)$ 在 y 的某邻域内单调可导, 且 $f'(y) \neq 0$ 则

$$[f^{-1}(x)]' = \frac{1}{f'(y)} \quad \text{或} \quad \frac{dy}{dx} = \frac{1}{\frac{dx}{dy}}$$

证: 在 x 处给增量 $\Delta x \neq 0$, 由反函数的单调性知

$$\Delta y = f^{-1}(x + \Delta x) - f^{-1}(x) \neq 0, \therefore \frac{\Delta y}{\Delta x} = \frac{1}{\frac{\Delta x}{\Delta y}}$$

且由反函数的连续性知 $\Delta x \rightarrow 0$ 时必有 $\Delta y \rightarrow 0$, 因此

$$[f^{-1}(x)]' = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta y \rightarrow 0} \frac{1}{\frac{\Delta x}{\Delta y}} = \frac{1}{f'(y)}$$



例3. 求反三角函数的导数.

解: 设 $y = \arcsin x$, 则 $x = \sin y$, $y \in (-\frac{\pi}{2}, \frac{\pi}{2})$,

$\therefore \cos y > 0$, 则

$$\begin{aligned} (\arcsin x)' &= \frac{1}{(\sin y)'} = \frac{1}{\cos y} = \frac{1}{\sqrt{1 - \sin^2 y}} \\ &= \frac{1}{\sqrt{1 - x^2}} \end{aligned}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1 - x^2}}$$

利用

$$\arccos x = \frac{\pi}{2} - \arcsin x$$

类似可求得

$$(\arctan x)' = \frac{1}{1 + x^2}, \quad (\operatorname{arccot} x)' = -\frac{1}{1 + x^2}$$



三、复合函数求导法则

定理3. $u = g(x)$ 在点 x 可导, $y = f(u)$ 在点 $u = g(x)$ 可导,
 \implies 复合函数 $y = f[g(x)]$ 在点 x 可导, 且

$$\frac{dy}{dx} = f'(u)g'(x) = f'(g(x))g'(x)$$

证: $\because y = f(u)$ 在点 u 可导, 故 $\lim_{\Delta u \rightarrow 0} \frac{\Delta y}{\Delta u} = f'(u)$

$$\therefore \Delta y = f'(u)\Delta u + \alpha\Delta u \quad (\text{当 } \Delta u \rightarrow 0 \text{ 时 } \alpha \rightarrow 0)$$

故有
$$\frac{\Delta y}{\Delta x} = f'(u)\frac{\Delta u}{\Delta x} + \alpha\frac{\Delta u}{\Delta x} \quad (\Delta x \neq 0)$$

$$\therefore \frac{dy}{dx} = \lim_{\Delta x \rightarrow 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \rightarrow 0} \left[f'(u)\frac{\Delta u}{\Delta x} + \alpha\frac{\Delta u}{\Delta x} \right] = f'(u)g'(x)$$

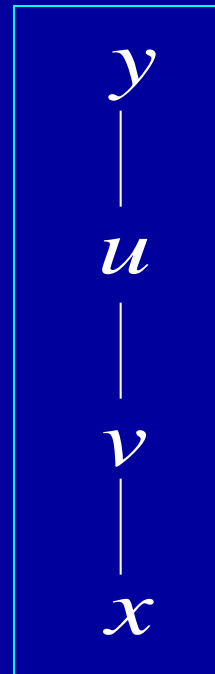


推广：此法则可推广到多个中间变量的情形.

例如, $y = f(u)$, $u = \varphi(v)$, $v = \psi(x)$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dv} \cdot \frac{dv}{dx}$$

$$= f'(u) \cdot \varphi'(v) \cdot \psi'(x)$$



关键：搞清复合函数结构, 由外向内逐层求导.



例4. 求下列导数: (1) $(x^\mu)'$; (2) $(x^x)'$;

解: (1) $(x^\mu)' = (e^{\mu \ln x})'$
 $= e^{\mu \ln x} \cdot (\mu \ln x)'$
 $= x^\mu \cdot \frac{\mu}{x}$
 $= \mu x^{\mu-1}$

(2) $(x^x)' = (e^{x \ln x})'$
 $= e^{x \ln x} \cdot (x \ln x)'$
 $= x^x (\ln x + 1)$



例5. 设 $y = \ln \cos(e^x)$, 求 $\frac{dy}{dx}$.

解: $\frac{dy}{dx} = \frac{1}{\cos(e^x)} \cdot (-\sin(e^x)) \cdot e^x = -e^x \tan(e^x)$

思考: 若 $f'(u)$ 存在, 如何求 $f(\ln \cos(e^x))$ 的导数?

$$\frac{df}{dx} = f'(\ln \cos(e^x)) \cdot (\ln \cos(e^x))' = \dots$$

这两个记号含义不同

$$f'(u) \Big|_{u=\ln \cos(e^x)}$$



例6. 设 $y = \ln(\underline{x} + \underline{\sqrt{x^2 + 1}})$, 求 y' .

解:
$$y' = \frac{1}{x + \sqrt{x^2 + 1}} \cdot \left(1 + \frac{1}{2\sqrt{x^2 + 1}} \cdot 2x \right)$$
$$= \frac{1}{\sqrt{x^2 + 1}}$$



四、初等函数的求导问题

1. 常数和基本初等函数的导数 (P92)

$$(C)' = 0$$

$$(\sin x)' = \cos x$$

$$(\tan x)' = \sec^2 x$$

$$(\sec x)' = \sec x \tan x$$

$$(a^x)' = a^x \ln a$$

$$(\log_a x)' = \frac{1}{x \ln a}$$

$$(\arcsin x)' = \frac{1}{\sqrt{1-x^2}}$$

$$(\arctan x)' = \frac{1}{1+x^2}$$

$$(x^\mu)' = \mu x^{\mu-1}$$

$$(\cos x)' = -\sin x$$

$$(\cot x)' = -\csc^2 x$$

$$(\csc x)' = -\csc x \cot x$$

$$(e^x)' = e^x$$

$$(\ln x)' = \frac{1}{x}$$

$$(\arccos x)' = -\frac{1}{\sqrt{1-x^2}}$$

$$(\operatorname{arccot} x)' = -\frac{1}{1+x^2}$$



2. 有限次四则运算的求导法则

$$(u \pm v)' = u' \pm v'$$

$$(uv)' = u'v + uv'$$

$$(Cu)' = Cu' \quad (C \text{ 为常数})$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2} \quad (v \neq 0)$$

3. 复合函数求导法则

$$y = f(u), u = \varphi(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = f'(u) \cdot \varphi'(x)$$

4. 初等函数在定义区间内可导, 且导数仍为初等函数

说明: 最基本的公式

$$(C)' = 0$$

$$(\sin x)' = \cos x$$

$$(\ln x)' = \frac{1}{x}$$

由定义证, 其他公式
用求导法则推出.



例7. $y = \frac{\sqrt{x+1} - \sqrt{x-1}}{\sqrt{x+1} + \sqrt{x-1}}$, 求 y' .

先化简后求导

解: $\because y = \frac{2x - 2\sqrt{x^2 - 1}}{2} = x - \sqrt{x^2 - 1}$

$\therefore y' = 1 - \frac{1}{2\sqrt{x^2 - 1}} \cdot (2x) = 1 - \frac{x}{\sqrt{x^2 - 1}}$

例8. 设 $y = x^{a^a} + a^{x^a} + a^{a^x}$ ($a > 0$), 求 y' .

解: $y' = a^a x^{a^a - 1} + a^{x^a} \ln a \cdot a x^{a-1}$
 $+ a^{a^x} \ln a \cdot a^x \ln a$



例9. $y = e^{\sin x^2} \arctan \sqrt{x^2 - 1}$, 求 y' .

解: $y' = (e^{\sin x^2} \cdot \cos x^2 \cdot 2x) \arctan \sqrt{x^2 - 1}$

$$+ e^{\sin x^2} \left(\frac{1}{x^2} \cdot \frac{1}{2\sqrt{x^2 - 1}} \cdot 2x \right)$$

$$= 2x \cos x^2 e^{\sin x^2} \arctan \sqrt{x^2 - 1} + \frac{1}{x\sqrt{x^2 - 1}} e^{\sin x^2}$$

关键: 搞清复合函数结构
由外向内逐层求导



例10. 设 $y = \frac{1}{2} \arctan \sqrt{1+x^2} + \frac{1}{4} \ln \frac{\sqrt{1+x^2}+1}{\sqrt{1+x^2}-1}$, 求 y' .

解:

$$\begin{aligned} y' &= \frac{1}{2} \frac{1}{1+(\sqrt{1+x^2})^2} \cdot \frac{x}{\sqrt{1+x^2}} \\ &\quad + \frac{1}{4} \left(\frac{1}{\sqrt{1+x^2}+1} \cdot \frac{x}{\sqrt{1+x^2}} - \frac{1}{\sqrt{1+x^2}-1} \cdot \frac{x}{\sqrt{1+x^2}} \right) \\ &= \frac{1}{2} \frac{x}{\sqrt{1+x^2}} \left(\frac{1}{2+x^2} - \frac{1}{x^2} \right) \\ &= \frac{-1}{(2x+x^3)\sqrt{1+x^2}} \end{aligned}$$



2.2 作业

P94-95

2(2), (8), (10);

3 (2), (3);

6 (6), (8);

7 (3), (7), (10);

8 (4), (5), (8), (10);

10;

11 (3), (8), (10);



内容小结

求导公式及求导法则 (见P95 ~ P96)

注意: 1) $(uv)' \neq u'v'$, $\left(\frac{u}{v}\right)' \neq \frac{u'}{v'}$

2) 理清复合函数结构, 由外向内逐层求导.

思考与练习

1. $\left(\frac{1}{\sqrt{x}\sqrt{x}}\right)' = \left(\left(\frac{1}{x}\right)^{\frac{3}{4}}\right)' \neq \frac{3}{4}\left(\frac{1}{x}\right)^{-\frac{1}{4}}$ 对吗?

\searrow

$= \frac{3}{4}\left(\frac{1}{x}\right)^{-\frac{1}{4}} \cdot \frac{-1}{x^2}$



2. 设 $f(x) = (x - a)\varphi(x)$, 其中 $\varphi(x)$ 在 $x = a$ 处连续,
在求 $f'(a)$ 时, 下列做法是否正确?

$$\begin{aligned} \text{因 } f'(x) &\neq \varphi(x) + (x - a)\varphi'(x) \\ \text{故 } f'(a) &= \varphi(a) \end{aligned}$$

正确解法: 由于 $f(a) = 0$, 故

$$\begin{aligned} f'(a) &= \lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a} = \lim_{x \rightarrow a} \frac{(x - a)\varphi(x)}{x - a} \\ &= \lim_{x \rightarrow a} \varphi(x) = \varphi(a) \end{aligned}$$



3. 求下列函数的导数

$$(1) \ y = \left(\frac{a}{x}\right)^b; \quad (2) \ y = \left(\frac{a}{b}\right)^{-x}.$$

解: (1) $y' = b \left(\frac{a}{x}\right)^{b-1} \cdot \left(-\frac{a}{x^2}\right) = -\frac{a^b b}{x^{b+1}}$

$$(2) \ y' = \left(\frac{a}{b}\right)^{-x} \ln \frac{a}{b} \cdot (-x)' = -\left(\frac{b}{a}\right)^x \ln \frac{a}{b}$$

或 $y' = \left(\left(\frac{b}{a}\right)^x\right)' = \left(\frac{b}{a}\right)^x \ln \frac{b}{a}$



4. 设 $f(x) = \underline{x(x-1)(x-2)\cdots(x-99)}$, 求 $f'(0)$.

解: 方法1 利用导数定义.

$$\begin{aligned} f'(0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0} (x-1)(x-2)\cdots(x-99) = -99! \end{aligned}$$

方法2 利用求导公式.

$$\begin{aligned} f'(x) &= (x)' \cdot [(x-1)(x-2)\cdots(x-99)] \\ &\quad + x \cdot [(x-1)(x-2)\cdots(x-99)]' \\ \therefore f'(0) &= -99! \end{aligned}$$



备用题 1. 设 $y = \cot \frac{\sqrt{x}}{2} + \tan \frac{2}{\sqrt{x}}$, 求 y' .

解:
$$y' = -\csc^2 \frac{\sqrt{x}}{2} \cdot \frac{1}{2} \frac{1}{2\sqrt{x}} + \sec^2 \frac{2}{\sqrt{x}} \cdot 2 \left(-\frac{1}{2} \frac{1}{\sqrt{x^3}} \right)$$
$$= -\frac{1}{4\sqrt{x}} \csc^2 \frac{\sqrt{x}}{2} - \frac{1}{\sqrt{x^3}} \sec^2 \frac{2}{\sqrt{x}}$$

2. 设 $y = f(f(f(x)))$, 其中 $f(x)$ 可导, 求 y' .

解: $y' = f'(f(f(x))) \cdot f'(f(x)) \cdot f'(x)$

