第二节

换元积分法

一、第一类换元法

二、第二类换元法

基本思路

设 $F'(u)=f(u)$, $u=\varphi(x)$ 可导，则有

$$\mathrm{d}F[\varphi(x)]=f[\varphi(x)]\varphi'(x)\mathrm{d}x$$

$$\therefore \quad \int f[\varphi(x)]\varphi'(x)\mathrm{d}x=F[\varphi(x)]+C=F(u)+C\Big|_{u=\varphi(x)}$$

$$=\int f(u)\mathrm{d}u\Big|_{u=\varphi(x)}$$

$$\int f[\varphi(x)]\varphi'(x)\,\mathrm{d}x \underset{\text{第二类换元法}}{\overset{\text{第一类换元法}}{\rightleftarrows}} \int f(u)\,\mathrm{d}u$$

一、第一类换元法

定理1. 设 $f(u)$ 有原函数，$u=\varphi(x)$ 可导，则有换元公式

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(u)\mathrm{d}u \Big|_{u=\varphi(x)}$$

即

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$$

(也称**配元法**，**凑微分法**)

例1. 求 $\int (ax+b)^m \mathrm{d}x \quad (m \neq -1).$

解: 令 $u = ax+b$, 则 $\mathrm{d}u = a\mathrm{d}x$, 故

$$\text{原式} = \int u^m \frac{1}{a} \mathrm{d}u = \frac{1}{a} \cdot \frac{1}{m+1} u^{m+1} + C$$

$$= \frac{1}{a(m+1)} (\underline{ax+b})^{m+1} + C$$

注意换回原变量

注: 当 $m = -1$ 时

$$\int \frac{\mathrm{d}x}{ax+b} = \frac{1}{a} \ln|ax+b| + C$$

例2. 求 $\int \frac{\mathrm{d}x}{a^2+x^2}$.

想到公式

$$\int \frac{\mathrm{d}u}{1+u^2} = \arctan u + C$$

解: $\int \frac{\mathrm{d}x}{a^2+x^2} = \frac{1}{a^2}\int \frac{\mathrm{d}x}{1+(\frac{x}{a})^2}$

令 $u=\frac{x}{a}$, 则 $\mathrm{d}u = \frac{1}{a}\mathrm{d}x$

$$= \frac{1}{a}\int \frac{\mathrm{d}u}{1+u^2} = \frac{1}{a}\arctan u + C$$

$$= \frac{1}{a}\arctan(\frac{x}{a}) + C$$

例3. 求 $\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} \ (a>0).$

解: $$\int \frac{\mathrm{d}x}{\sqrt{a^2-x^2}} = \int \frac{\mathrm{d}\,x}{a\sqrt{1-(\frac{x}{a})^2}} = \int \frac{\mathrm{d}(\frac{x}{a})}{\sqrt{1-(\frac{x}{a})^2}}$$

$$= \arcsin\frac{x}{a} + C$$

想到 $$\int \frac{\mathrm{d}\,u}{\sqrt{1-u^2}} = \arcsin u + C$$

$$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x = \int f(\varphi(x))\mathrm{d}\varphi(x)$$ (直接配元)

例4. 求 $\int \tan x \mathrm{d}x$.

解: $\int \tan x \mathrm{d}x = \int \frac{\sin x}{\cos x} \mathrm{d}x = -\int \frac{\mathrm{d}\cos x}{\cos x}$

$$= -\ln|\cos x| + C$$

类似

$$\int \cot x \mathrm{d}x = \int \frac{\cos x \, \mathrm{d}x}{\sin x} = \int \frac{\mathrm{d}\sin x}{\sin x}$$

$$= \ln|\sin x| + C$$

例5. 求 $\int \frac{\mathrm{d}x}{x^2-a^2}$.

解:

$$\because \frac{1}{x^2-a^2}=\frac{1}{2a}\frac{(x+a)-(x-a)}{(x-a)(x+a)}=\frac{1}{2a}\left(\frac{1}{x-a}-\frac{1}{x+a}\right)$$

$$\therefore \text{原式}=\frac{1}{2a}\left[\int\frac{\mathrm{d}x}{x-a}-\int\frac{\mathrm{d}x}{x+a}\right]$$

$$=\frac{1}{2a}\left[\int\frac{\mathrm{d}(x-a)}{x-a}-\int\frac{\mathrm{d}(x+a)}{x+a}\right]$$

$$=\frac{1}{2a}\left[\ln|x-a|-\ln|x+a|\right]+C=\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|+C$$

常用的几种配元形式:

$$1)\ \int f(ax+b)\,\mathrm{d}x=\frac{1}{a}\int f(ax+b)\,\mathrm{d}(ax+b)$$

$$\left.\begin{array}{l}2)\ \displaystyle\int f(x^n)\,x^{n-1}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\mathrm{d}x^n \\[2ex] 3)\ \displaystyle\int f(x^n)\frac{1}{x}\,\mathrm{d}x=\frac{1}{n}\int f(x^n)\,\frac{1}{x^n}\,\mathrm{d}x^n\end{array}\right\}\text{万能凑幂法}$$

$$4)\ \int f(\sin x)\cos x\,\mathrm{d}x=\int f(\sin x)\,\mathrm{d}\sin x$$

$$5)\ \int f(\cos x)\sin x\,\mathrm{d}x=-\int f(\cos x)\,\mathrm{d}\cos x$$

6) $\int f(\tan x)\sec^2 x\mathrm{d}x = \int f(\tan x)\,\mathrm{d}\tan x$

7) $\int f(\mathrm{e}^x)\mathrm{e}^x\mathrm{d}x = \int f(\mathrm{e}^x)\,\mathrm{d}\mathrm{e}^x$

8) $\int f(\ln x)\frac{1}{x}\mathrm{d}x = \int f(\ln x)\,\mathrm{d}\ln x$

例6. 求 $\int \frac{\mathrm{d}x}{x(1+2\ln x)}$.

解: 原式 $= \int \frac{\mathrm{d}\ln x}{1+2\ln x} = \frac{1}{2}\int \frac{\mathrm{d}(1+2\ln x)}{1+2\ln x}$

$$= \frac{1}{2}\ln\left|1+2\ln x\right| + C$$

例7. 求 $\int \frac{e^{3\sqrt{x}}}{\sqrt{x}} dx$.

解: 原式 = $2\int e^{3\sqrt{x}} d\sqrt{x} = \frac{2}{3}\int e^{3\sqrt{x}} d(3\sqrt{x})$

$= \frac{2}{3} e^{3\sqrt{x}} + C$

例8. 求 $\int \sec^6 x dx$.

解: 原式 = $\int (\tan^2 x + 1)^2 \cdot d\tan x$

$= \int (\tan^4 x + 2\tan^2 x + 1) d\tan x$

$= \frac{1}{5}\tan^5 x + \frac{2}{3}\tan^3 x + \tan x + C$

例9. 求 $\int \frac{\mathrm{d}x}{1+\mathrm{e}^x}$.

解法1

$$\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{(1+\mathrm{e}^x)-\mathrm{e}^x}{1+\mathrm{e}^x}\,\mathrm{d}x = \int \mathrm{d}x - \int \frac{\mathrm{d}(1+\mathrm{e}^x)}{1+\mathrm{e}^x}$$

$$= x - \ln(1+\mathrm{e}^x) + C$$

解法2

$$\int \frac{\mathrm{d}x}{1+\mathrm{e}^x} = \int \frac{\mathrm{e}^{-x}}{1+\mathrm{e}^{-x}}\,\mathrm{d}x = -\int \frac{\mathrm{d}(1+\mathrm{e}^{-x})}{1+\mathrm{e}^{-x}}$$

$$= -\ln(1+\mathrm{e}^{-x}) + C$$

$$-\ln(1+\mathrm{e}^{-x}) = -\ln[\mathrm{e}^{-x}(\mathrm{e}^x+1)]$$ 两法结果一样

例10. 求 $\int \sec x \mathrm{d}x$.

解法1

$$\int \sec x \mathrm{d}x = \int \frac{\cos x}{\cos^2 x} \mathrm{d}x = \int \frac{\mathrm{d}\sin x}{1-\sin^2 x}$$

$$= \frac{1}{2} \int \left[\frac{1}{1+\sin x} + \frac{1}{1-\sin x} \right] \mathrm{d}\sin x$$

$$= \frac{1}{2} \left[\ln|1+\sin x| - \ln|1-\sin x| \right] + C$$

$$= \frac{1}{2} \ln\left| \frac{1+\sin x}{1-\sin x} \right| + C$$

解法 2

$$\int \sec x \mathrm{d}x = \int \frac{\sec x\,(\sec x + \tan x)}{\sec x + \tan x}\mathrm{d}x$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\mathrm{d}x$$

$$= \int \frac{\mathrm{d}(\sec x + \tan x)}{\sec x + \tan x}$$

$$= \ln\left|\ \sec x + \tan x\ \right| + C$$

同样可证

$$\int \csc x \mathrm{d}x = \ln\left|\csc x - \cot x\right| + C$$

或 $$\int \csc x \mathrm{d}x = \ln\left|\tan\frac{x}{2}\right| + C$$ (P199 例18)

例11. 求 $\displaystyle\int \frac{x^3}{(x^2+a^2)^{3/2}}\,\mathrm{d}x$.

解: 原式 $\displaystyle = \frac{1}{2}\int \frac{x^2\,\mathrm{d}x^2}{(x^2+a^2)^{3/2}} = \frac{1}{2}\int \frac{(x^2+a^2)-a^2}{(x^2+a^2)^{3/2}}\,\mathrm{d}x^2$

$$= \frac{1}{2}\int (x^2+a^2)^{-1/2}\,\mathrm{d}(x^2+a^2)$$

$$-\frac{a^2}{2}\int (x^2+a^2)^{-3/2}\,\mathrm{d}(x^2+a^2)$$

$$= \sqrt{x^2+a^2} + \frac{a^2}{\sqrt{x^2+a^2}} + C$$

例12. 求 $\int \cos^4 x\,\mathrm{d}x$.

解: $\because \cos^4 x = (\cos^2 x)^2 = (\frac{1+\cos 2x}{2})^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}(1 + 2\cos 2x + \frac{1+\cos 4x}{2})$$

$$= \frac{1}{4}(\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)$$

$$\therefore \quad \int \cos^4 x\,\mathrm{d}x = \frac{1}{4}\int (\frac{3}{2} + 2\cos 2x + \frac{1}{2}\cos 4x)\,\mathrm{d}x$$

$$= \frac{1}{4}\left[\frac{3}{2}\int \mathrm{d}x + \int \cos 2x\,\mathrm{d}(2x) + \frac{1}{8}\int \cos 4x\,\mathrm{d}(4x)\right]$$

$$= \frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x + C$$

例13. 求 $\int \sin^2 x \cos^2 3x \,\mathrm{d}x$.

解: $\because \sin^2 x \cos^2 3x = [\frac{1}{2}(\sin 4x - \sin 2x)]^2$

$$= \frac{1}{4}\sin^2 4x - \frac{1}{4}\cdot 2\sin 4x \sin 2x + \frac{1}{4}\sin^2 2x$$

$$= \frac{1}{8}(1-\cos 8x) - \sin^2 2x \cos 2x + \frac{1}{8}(1-\cos 4x)$$

$\therefore$ 原式 $= \frac{1}{4}\int \mathrm{d}x - \frac{1}{64}\int \cos 8x \,\mathrm{d}(8x)$

$$- \frac{1}{2}\int \sin^2 2x \,\mathrm{d}(\sin 2x) - \frac{1}{32}\int \cos 4x \,\mathrm{d}(4x)$$

$$= \frac{1}{4}x - \frac{1}{64}\sin 8x - \frac{1}{6}\sin^3 2x - \frac{1}{32}\sin 4x + C$$

例14. 求 $\int \frac{x+1}{x(1+x\mathrm{e}^x)}\mathrm{d}x$.

解: 原式 $= \int \frac{(x+1)\ \mathrm{e}^x}{x\mathrm{e}^x(1+x\mathrm{e}^x)}\mathrm{d}x = \int\left(\frac{1}{x\mathrm{e}^x} - \frac{1}{1+x\mathrm{e}^x}\right)\mathrm{d}(x\mathrm{e}^x)$

$$= \ln\left|x\mathrm{e}^x\right| - \ln\left|1+x\mathrm{e}^x\right| + C$$

$$= x + \ln|x| - \ln\left|1+x\mathrm{e}^x\right| + C$$

分析: $\frac{1}{x\mathrm{e}^x(1+x\mathrm{e}^x)} = \frac{1+x\mathrm{e}^x - x\mathrm{e}^x}{x\mathrm{e}^x(1+x\mathrm{e}^x)} = \frac{1}{x\mathrm{e}^x} - \frac{1}{1+x\mathrm{e}^x}$

$$(x+1)\mathrm{e}^x\mathrm{d}x = x\mathrm{e}^x\mathrm{d}x + \mathrm{e}^x\mathrm{d}x = \mathrm{d}(x\mathrm{e}^x)$$

例15. 求 $\int\left[\frac{f(x)}{f'(x)}-\frac{f''(x)f^2(x)}{f'^3(x)}\right]\mathrm{d}x.$

解：原式 $=\int\frac{f(x)}{f'(x)}\left[1-\frac{f''(x)f(x)}{f'^2(x)}\right]\mathrm{d}x$

$$=\int\frac{f(x)}{f'(x)}\cdot\frac{f'^2(x)-f''(x)f(x)}{f'^2(x)}\mathrm{d}x$$

$$=\int\frac{f(x)}{f'(x)}\,\mathrm{d}\left(\frac{f(x)}{f'(x)}\right)$$

$$=\frac{1}{2}\left[\frac{f(x)}{f'(x)}\right]^2+C$$

小结 常用简化技巧:

(1) 分项积分: 利用积化和差; 分式分项;

$$1 = \sin^2 x + \cos^2 x \text{等}$$

(2) 降低幂次: 利用倍角公式 , 如

$$\cos^2 x = \tfrac{1}{2}(1+\cos 2x)\,; \qquad \sin^2 x = \tfrac{1}{2}(1-\cos 2x)\,;$$

$$\text{万能凑幂法}\begin{cases} \int f(x^n)x^{n-1}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\,\mathrm{d}\,x^n \\ \int f(x^n)\frac{1}{x}\,\mathrm{d}x = \frac{1}{n}\int f(x^n)\frac{1}{x^n}\,\mathrm{d}\,x^n \end{cases}$$

(3) 统一函数: 利用三角公式 ; 配元方法

(4) 巧妙换元或配元

二、第二类换元法

➢第一类换元法解决的问题

$$\underset{\text{难求}}{\int f[\varphi(x)]\varphi'(x)\mathrm{d}x} = \underset{\text{易求}}{\int f(u)\mathrm{d}u}\Big|_{u=\varphi(x)}$$

➢若所求积分 $\int f(u)\mathrm{d}u$ 难求，$\int f[\varphi(x)]\varphi'(x)\mathrm{d}x$ 易求，

则得第二类换元积分法 .

定理2. 设 $x=\psi(t)$ 是单调可导函数，且 $\psi'(t)\neq 0$，

$f[\psi(t)]\psi'(t)$ 具有原函数，则有换元公式

$$\int f(x)\,\mathrm{d}x=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

其中 $t=\psi^{-1}(x)$ 是 $x=\psi(t)$ 的反函数.

证: 设 $f[\psi(t)]\psi'(t)$ 的原函数为 $\Phi(t)$，令

$$F(x)=\Phi[\psi^{-1}(x)]$$

则

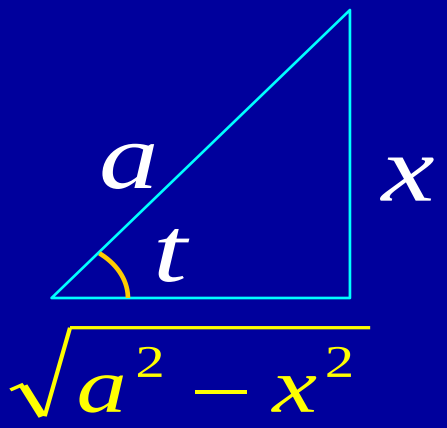
$$F'(x)=\frac{\mathrm{d}\Phi}{\mathrm{d}t}\cdot\frac{\mathrm{d}t}{\mathrm{d}x}=f[\psi(t)]\cancel{\psi'(t)}\cdot\frac{1}{\cancel{\psi'(t)}}=f(x)$$

$$\therefore\quad \int f(x)\,\mathrm{d}x=F(x)+C=\Phi[\psi^{-1}(x)]+C$$

$$=\int f[\psi(t)]\psi'(t)\,\mathrm{d}t\Big|_{t=\psi^{-1}(x)}$$

例16. 求 $\int\sqrt{a^2-x^2}\,\mathrm{d}x\quad(a>0)$.

解: 令 $x=a\sin t,\ \ t\in(-\frac{\pi}{2},\frac{\pi}{2})$, 则



$$\sqrt{a^2-x^2}=\sqrt{a^2-a^2\sin^2 t}=a\cos t$$

$$\mathrm{d}x=a\cos t\,\mathrm{d}t$$

$$\therefore\ 原式\ =\int a\cos t\cdot a\cos t\,\mathrm{d}t=a^2\int\cos^2 t\,\mathrm{d}t$$

$$=a^2\int\frac{1+\cos 2t}{2}\mathrm{d}t\ =a^2\left(\frac{t}{2}+\frac{\sin 2t}{4}\right)+C$$

$$\sin 2t=2\sin t\cos t=2\cdot\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}$$

$$=\frac{a^2}{2}\arcsin\frac{x}{a}+\frac{1}{2}x\sqrt{a^2-x^2}+C$$

例17. 求 $\int \frac{\mathrm{d}x}{\sqrt{x^2+a^2}} \quad (a>0).$

解: 令 $x = a\tan t,\ t\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, 则

$$\sqrt{x^2+a^2} = \sqrt{a^2\tan^2 t + a^2} = a\sec t$$

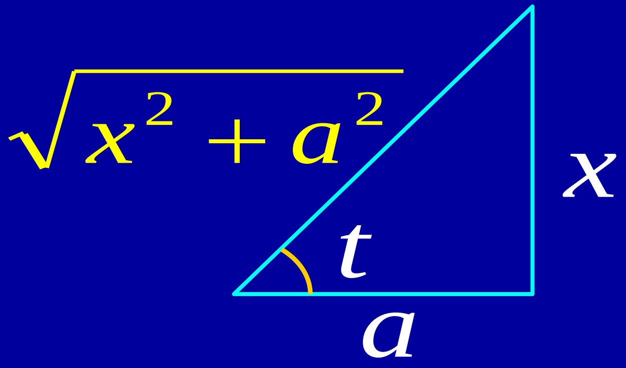
$$\mathrm{d}x = a\sec^2 t\,\mathrm{d}t$$

$$\therefore \text{原式} = \int \frac{a\sec^2 t}{a\sec t}\mathrm{d}t = \int \sec t\,\mathrm{d}t$$

$$= \ln\left|\sec t + \tan t\right| + C_1$$

$$= \ln\left[\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right] + C_1$$

$$= \ln\left[x + \sqrt{x^2+a^2}\right] + C \quad (C = C_1 - \ln a)$$



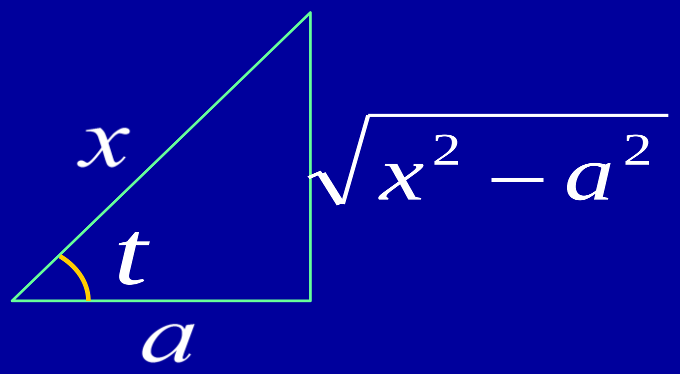
例18. 求 $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}}\quad (a>0).$

解: 当 $x>a$ 时，令 $x=a\sec t,\ t\in(0,\frac{\pi}{2})$，则

$$\sqrt{x^2-a^2}=\sqrt{a^2\sec^2 t-a^2}=a\tan t$$

$$\mathrm{d}x=a\sec t\tan t\,\mathrm{d}t$$

$$\therefore \text{原式} =\int\frac{a\sec t\tan t}{a\tan t}\mathrm{d}t=\int\sec t\,\mathrm{d}t$$



$$=\ln\left|\sec t+\tan t\right|+C_1$$

$$=\ln\left|\frac{x}{a}+\frac{\sqrt{x^2-a^2}}{a}\right|+C_1$$

$$=\ln\left|x+\sqrt{x^2-a^2}\right|+C\qquad (C=C_1-\ln a)$$

当$x<-a$时，令 $x=-u$，则$u>a$，于是

$$\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = -\int \frac{\mathrm{d}u}{\sqrt{u^2-a^2}} = -\ln\left|u+\sqrt{u^2-a^2}\right| + C_1$$

$$= -\ln\left|-x+\sqrt{x^2-a^2}\right| + C_1$$

$$= -\ln\left|\frac{a^2}{-x-\sqrt{x^2-a^2}}\right| + C_1$$

$$= \ln\left|x+\sqrt{x^2-a^2}\right| + C \quad (C = C_1 - 2\ln a)$$

$x>a$时，$\displaystyle\int \frac{\mathrm{d}x}{\sqrt{x^2-a^2}} = \ln\left|x+\sqrt{x^2-a^2}\right| + C$

例19. 求 $\int \frac{\sqrt{a^2-x^2}}{x^4}\,\mathrm{d}x$.

解: 令 $x=\frac{1}{t}$,则 $\mathrm{d}x=\frac{-1}{t^2}\,\mathrm{d}\,t$

原式 $=\int \frac{\sqrt{a^2-\frac{1}{t^2}}}{\frac{1}{t^4}}\cdot\frac{-1}{t^2}\,\mathrm{d}\,t=-\int (a^2t^2-1)^{\frac{1}{2}}\,|\,t\,|\,\mathrm{d}\,t$

当 $x>0$时,

原式 $=-\frac{1}{2a^2}\int (a^2t^2-1)^{\frac{1}{2}}\,\mathrm{d}(a^2t^2-1)$

$=-\frac{(a^2t^2-1)^{\frac{3}{2}}}{3a^2}+C=-\frac{(a^2-x^2)^{\frac{3}{2}}}{3a^2x^3}+C$

当 $x<0$ 时, 类似可得同样结果 .

小结:

1. 第二类换元法常见类型:

1) $\int f(x, \sqrt[n]{ax+b})\,\mathrm{d}x$, 令 $t=\sqrt[n]{ax+b}$

2) $\int f(x, \sqrt[n]{\frac{ax+b}{cx+d}})\,\mathrm{d}x$, 令 $t=\sqrt[n]{\frac{ax+b}{cx+d}}$

$\Bigg\}$ 第四节讲

3) $\int f(x, \sqrt{a^2-x^2})\,\mathrm{d}x$, 令 $x=a\sin t$ 或 $x=a\cos t$

4) $\int f(x, \sqrt{a^2+x^2})\,\mathrm{d}x$, 令 $x=a\tan t$ 或 $x=a\,\mathrm{sh}\,t$

5) $\int f(x, \sqrt{x^2-a^2})\,\mathrm{d}x$, 令 $x=a\sec t$ 或 $x=a\,\mathrm{ch}\,t$

6) $\int f(a^x)\,\mathrm{d}x$ ， 令 $t = a^x$

7) 分母中因子次数较高时, 可试用**倒代换**

2. 常用基本积分公式的补充 (P205 ~ P206)

(16) $\int \tan x \,\mathrm{d}\, x = -\ln\left|\cos x\right| + C$

(17) $\int \cot x \mathrm{d}x = \ln\left|\sin x\right| + C$

(18) $\int \sec x \mathrm{d}x = \ln\left|\sec x + \tan x\right| + C$

(19) $\int \csc x \mathrm{d}x = \ln\left|\csc x - \cot x\right| + C$

(20) $$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\arctan\frac{x}{a} + C$$

(21) $$\int \frac{1}{x^2 - a^2}\,\mathrm{d}x = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

(22) $$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \arcsin\frac{x}{a} + C$$

(23) $$\int \frac{1}{\sqrt{x^2 + a^2}}\,\mathrm{d}x = \ln(x + \sqrt{x^2 + a^2}) + C$$

(24) $$\int \frac{1}{\sqrt{x^2 - a^2}}\,\mathrm{d}x = \ln\left|\, x + \sqrt{x^2 - a^2}\,\right| + C$$

例20. 求 $\int \frac{\mathrm{d}x}{x^2+2x+3}$.

解: 原式 $= \int \frac{1}{(x+1)^2+(\sqrt{2})^2}\mathrm{d}(x+1)$

$$= \frac{1}{\sqrt{2}}\arctan\frac{x+1}{\sqrt{2}}+C$$ (P205 公式 (20))

例21. 求 $I = \int \frac{\mathrm{d}x}{\sqrt{4x^2+9}}$.

解: $I = \frac{1}{2}\int \frac{\mathrm{d}(2x)}{\sqrt{(2x)^2+3^2}} = \frac{1}{2}\ln\left|2x+\sqrt{4x^2+9}\right|+C$

(P205 公式 (23))

例22. 求 $\int \frac{\mathrm{d}x}{\sqrt{1+x-x^2}}$.

解: 原式 $= \int \frac{\mathrm{d}(x-\frac{1}{2})}{\sqrt{(\frac{\sqrt{5}}{2})^2-(x-\frac{1}{2})^2}} = \arcsin\frac{2x-1}{\sqrt{5}} + C$

(P205 公式 (22))

例23. 求 $\int \frac{\mathrm{d}x}{\sqrt{e^{2x}-1}}$.

解: 原式 $= -\int \frac{\mathrm{d}\,e^{-x}}{\sqrt{1-e^{-2x}}} = -\arcsin e^{-x} + C$

(P205 公式 (22))

例24. 求 $\int \frac{\mathrm{d}x}{x^2\sqrt{x^2+a^2}}$.

解: 令 $x=\frac{1}{t}$,得

原式 $=-\int \frac{t}{\sqrt{a^2t^2+1}}\,\mathrm{d}t$

$$=-\frac{1}{2a^2}\int \frac{\mathrm{d}(a^2t^2+1)}{\sqrt{a^2t^2+1}}=-\frac{1}{a^2}\sqrt{a^2t^2+1}+C$$

$$=-\frac{\sqrt{x^2+a^2}}{a^2x}+C$$

例25. 求 $\displaystyle\int \frac{\mathrm{d}x}{(x+1)^3\sqrt{x^2+2x}}$.

解: 原式 $\displaystyle = \int \frac{\mathrm{d}x}{(x+1)^3\sqrt{(x+1)^2-1}}$

$\boxed{\text{令}\quad x+1=\frac{1}{t}}$

$$= \int \frac{t^3}{\sqrt{\frac{1}{t^2}-1}}\left(-\frac{1}{t^2}\right)\mathrm{d}t = -\int \frac{t^2}{\sqrt{1-t^2}}\,\mathrm{d}t$$

$$= \int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\,\mathrm{d}t = \boxed{\int \sqrt{1-t^2}\,\mathrm{d}t} - \int \frac{1}{\sqrt{1-t^2}}\,\mathrm{d}t$$

$$= \frac{1}{2}t\sqrt{1-t^2} + \frac{1}{2}\arcsin t - \arcsin t + C$$

例16

$$= \frac{1}{2}\frac{\sqrt{x^2+2x}}{(x+1)^2} - \frac{1}{2}\arcsin\frac{1}{x+1} + C$$

4.2 作 业

P207-208

2 (4) , (5) , (9) , (11) , (12) , (16) , (20) ,

(21) , (23) , (28) , (29) , (30) , (32) , (33) ,

(35) , (36) , (38), (40) , (42) , (44)

思考与练习1

1. 下列各题求积方法有何不同?

$$(1)\ \int \frac{\mathrm{d}x}{4+x} = \int \frac{\mathrm{d}(4+x)}{4+x}$$

$$(2)\ \int \frac{\mathrm{d}x}{4+x^2} = \frac{1}{2}\int \frac{\mathrm{d}(\frac{x}{2})}{1+(\frac{x}{2})^2}$$

$$(3)\ \int \frac{x}{4+x^2}\,\mathrm{d}x = \frac{1}{2}\int \frac{\mathrm{d}(4+x^2)}{4+x^2}$$

$$(4)\ \int \frac{x^2}{4+x^2}\,\mathrm{d}x = \int \left[1-\frac{4}{4+x^2}\right]\mathrm{d}\,x$$

$$(5)\ \int \frac{\mathrm{d}x}{4-x^2} = \frac{1}{4}\int \left[\frac{1}{2-x}+\frac{1}{2+x}\right]\mathrm{d}x$$

$$(6)\ \int \frac{\mathrm{d}x}{\sqrt{4x-x^2}} = \int \frac{\mathrm{d}(x-2)}{\sqrt{4-(x-2)^2}}$$

2. 求$\int \frac{\mathrm{d}x}{x(x^{10}+1)}$.

提示:

法1 $$\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{(x^{10}+1)-x^{10}}{x(x^{10}+1)}\mathrm{d}x$$

法2 $$\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \frac{1}{10}\int \frac{\mathrm{d}\,x^{10}}{x^{10}(x^{10}+1)}$$

法3 $$\int \frac{\mathrm{d}x}{x(x^{10}+1)} = \int \frac{\mathrm{d}x}{x^{11}(1+x^{-10})} = \frac{-1}{10}\int \frac{\mathrm{d}\,x^{-10}}{1+x^{-10}}$$

思考与练习2

1. 下列积分应如何换元才使积分简便？

(1) $\int \frac{x^5}{\sqrt{1+x^2}}\,\mathrm{d}x$

令 $t=\sqrt{1+x^2}$

(2) $\int \frac{\mathrm{d}x}{\sqrt{1+\mathrm{e}^x}}$

令 $t=\sqrt{1+\mathrm{e}^x}$

(3) $\int \frac{\mathrm{d}x}{x(x^7+2)}$

令 $t=\frac{1}{x}$

2. 已知 $\int x^5 f(x)\,\mathrm{d}x = \sqrt{x^2-1} + C$，求 $\int f(x)\,\mathrm{d}x$.

解: 两边求导, 得 $x^5 f(x) = \dfrac{x}{\sqrt{x^2-1}}$，则

$$\int f(x)\,\mathrm{d}x = \int \frac{\mathrm{d}x}{x^4\sqrt{x^2-1}} \quad (\text{令 } t = \frac{1}{x})$$

$$= \int \frac{-t^3\,\mathrm{d}t}{\sqrt{1-t^2}} = \frac{1}{2}\int \frac{(1-t^2)-1}{\sqrt{1-t^2}}\,\mathrm{d}t^2$$

$$= \frac{-1}{2}\int (1-t^2)^{\frac{1}{2}}\,\mathrm{d}(1-t^2) + \frac{1}{2}\int (1-t^2)^{-\frac{1}{2}}\,\mathrm{d}(1-t^2)$$

$$= \frac{-1}{3}(1-t^2)^{\frac{3}{2}} + (1-t^2)^{\frac{1}{2}} + C = \cdots$$

(代回原变量)

备用题 1. 求下列积分:

1) $$\int x^2 \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}x = \frac{1}{3}\int \frac{1}{\sqrt{x^3+1}}\,\mathrm{d}(x^3+1)$$

$$= \frac{2}{3}\sqrt{x^3+1}+C$$

2) $$\int \frac{2x+3}{\sqrt{1+2x-x^2}}\,\mathrm{d}x = \int \frac{-(2-2x)+5}{\sqrt{1+2x-x^2}}\,\mathrm{d}x$$

$$= -\int \frac{\mathrm{d}(1+2x-x^2)}{\sqrt{1+2x-x^2}} + 5\int \frac{\mathrm{d}(x-1)}{\sqrt{2-(x-1)^2}}$$

$$= -2\sqrt{1+2x-x^2} + 5\arcsin\frac{x-1}{\sqrt{2}} + C$$

2. 求不定积分 $\int \frac{2\sin x \cos x \sqrt{1+\sin^2 x}}{2+\sin^2 x} \mathrm{d}x.$

解: 利用凑微分法，得

$$\text{原式} = \int \frac{\sqrt{1+\sin^2 x}}{2+\sin^2 x} \mathrm{d}(1+\sin^2 x)$$

令 $t = \sqrt{1+\sin^2 x}$

$$= \int \frac{2t^2}{1+t^2} \mathrm{d}t = 2\int (1-\frac{1}{1+t^2}) \mathrm{d}t$$

$$= 2t - 2\arctan t + C$$

$$= 2[\sqrt{1+\sin^2 x} - \arctan\sqrt{1+\sin^2 x}] + C$$

3. 求不定积分 $\int \frac{1}{(1+x^2)\sqrt{1-x^2}} \mathrm{d}x.$

解: 令 $x = \sin t$, $1+x^2 = 1+\sin^2 t$, $\mathrm{d}x = \cos t \mathrm{d}t$

原式 $= \int \frac{\cos t}{(1+\sin^2 t)\cos t} \mathrm{d}t = \int \frac{1}{1+\sin^2 t} \mathrm{d}t$

分子分母同除以 $\cos^2 t$

$= \int \frac{\sec^2 t}{\sec^2 t + \tan^2 t} \mathrm{d}t = \int \frac{1}{1+2\tan^2 t} \mathrm{d}\tan t$

$= \frac{1}{\sqrt{2}} \int \frac{1}{1+(\sqrt{2}\tan t)^2} \mathrm{d}\sqrt{2}\tan t$

$= \frac{1}{\sqrt{2}} \arctan(\sqrt{2}\tan t) + C = \frac{1}{\sqrt{2}} \arctan \frac{\sqrt{2}x}{\sqrt{1-x^2}} + C$