

第三节

定积分的换元法和

分部积分法

不定积分 $\begin{cases} \text{换元积分法} \\ \text{分部积分法} \end{cases} \longrightarrow$ 定积分 $\begin{cases} \text{换元积分法} \\ \text{分部积分法} \end{cases}$



一、定积分的换元法

二、定积分的分部积分法



一、定积分的换元法

定理1. 设函数 $f(x) \in C[a, b]$, 函数 $x = \varphi(t)$ 满足:

1) $\varphi(t) \in C^1[\alpha, \beta], \varphi(\alpha) = a, \varphi(\beta) = b;$

2) 在 $[\alpha, \beta]$ 上 $a \leq \varphi(t) \leq b,$

则
$$\int_a^b f(x) dx = \int_{\alpha}^{\beta} f[\varphi(t)] \varphi'(t) dt$$

证: 所证等式两边被积函数都连续, 因此积分都存在, 且它们的原函数也存在. 设 $F(x)$ 是 $f(x)$ 的一个原函数,

则 $F[\varphi(t)]$ 是 $f[\varphi(t)] \varphi'(t)$ 的原函数, 因此有

$$\begin{aligned} \int_a^b f(x) dx &= F(b) - F(a) = F[\varphi(\beta)] - F[\varphi(\alpha)] \\ &= \int_{\alpha}^{\beta} f[\varphi(t)] \varphi'(t) dt \end{aligned}$$



$$\int_a^b f(x) dx = \int_\alpha^\beta f[\varphi(t)] \varphi'(t) dt$$

说明:

- 1) 当 $\beta < \alpha$, 即区间换为 $[\beta, \alpha]$ 时, 定理 1 仍成立.
- 2) 必需注意**换元必换限**, 原函数中的变量不必代回.
- 3) 换元公式也可反过来使用, 即

$$\int_\alpha^\beta f[\varphi(t)] \varphi'(t) dt = \int_a^b f(x) dx \quad (\text{令 } x = \varphi(t))$$

或配元 $\int_\alpha^\beta f[\varphi(t)] \underline{\varphi'(t)} dt = \int_\alpha^\beta f[\varphi(t)] d\varphi(t)$

配元不换限

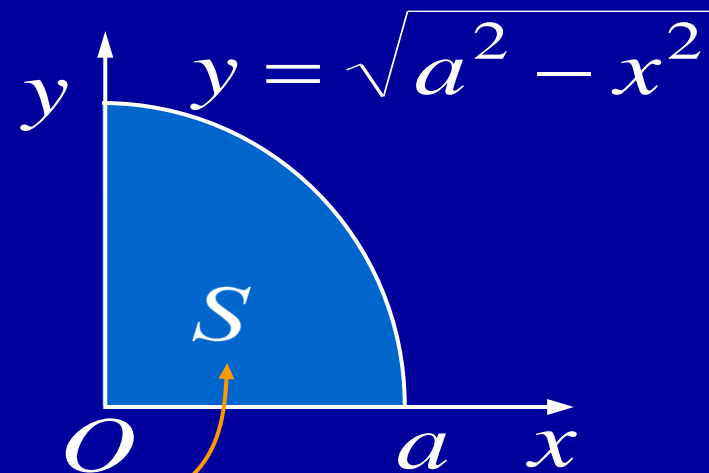


例1. 计算 $\int_0^a \sqrt{a^2 - x^2} \, dx \quad (a > 0).$

解: 令 $x = a \sin t$, 则 $dx = a \cos t \, dt$, 且

当 $x=0$ 时, $t=0$; $x=a$ 时, $t = \frac{\pi}{2}$.

$$\begin{aligned} \therefore \text{原式} &= a^2 \int_0^{\frac{\pi}{2}} \cos^2 t \, dt \\ &= \frac{a^2}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2t) \, dt \\ &= \frac{a^2}{2} \left(t + \frac{1}{2} \sin 2t \right) \bigg|_0^{\frac{\pi}{2}} = \frac{\pi a^2}{4} \end{aligned}$$



例2. 计算 $\int_0^4 \frac{x+2}{\sqrt{2x+1}} dx$.

解: 令 $t = \sqrt{2x+1}$, 则 $x = \frac{t^2-1}{2}$, $dx = t dt$, 且

当 $x=0$ 时, $t=1$; $x=4$ 时, $t=3$.

$$\therefore \text{原式} = \int_1^3 \frac{\frac{t^2-1}{2} + 2}{t} t dt$$

$$= \frac{1}{2} \int_1^3 (t^2 + 3) dt$$

$$= \frac{1}{2} \left(\frac{1}{3} t^3 + 3t \right) \bigg|_1^3 = \frac{22}{3}$$



例3. 设 $f(x) \in C[-a, a]$,

(1) 若 $f(-x) = f(x)$, 则 $\int_{-a}^a f(x) dx = 2 \int_0^a f(x) dx$

(2) 若 $f(-x) = -f(x)$, 则 $\int_{-a}^a f(x) dx = 0$

证: $\int_{-a}^a f(x) dx = \int_{-a}^0 f(x) dx + \int_0^a f(x) dx$

$$= \int_0^a f(-t) dt + \int_0^a f(x) dx$$

令 $x = -t$

$$= \int_0^a [f(-x) + f(x)] dx$$

$$= \begin{cases} 2 \int_0^a f(x) dx, & f(-x) = f(x) \text{ 时} \\ 0, & f(-x) = -f(x) \text{ 时} \end{cases}$$



例4. 设 $f(x)$ 是连续的周期函数, 周期为 T , 证明:

$$(1) \int_a^{a+T} f(x) \mathrm{d}x = \int_0^T f(x) \mathrm{d}x$$

$$(2) \int_a^{a+nT} f(x) \mathrm{d}x = n \int_0^T f(x) \mathrm{d}x \quad (n \in \mathbf{N}), \text{ 并由此计算}$$

$$I = \int_0^{n\pi} \sqrt{1 + \sin 2x} \mathrm{d}x$$

解: (1) 记 $\Phi(a) = \int_a^{a+T} f(x) \mathrm{d}x$, 则

$$\Phi'(a) = f(a+T) - f(a) = 0$$

可见 $\Phi(a)$ 与 a 无关, 因此 $\Phi(a) = \Phi(0)$, 即

$$\int_a^{a+T} f(x) \mathrm{d}x = \int_0^T f(x) \mathrm{d}x$$



$$(2) \int_a^{a+nT} f(x) dx = n \int_0^T f(x) dx \quad (n \in \mathbf{N}), \text{ 并由此计算}$$

$$\int_0^{n\pi} \sqrt{1 + \sin 2x} dx$$

$$(2) \int_a^{a+nT} f(x) dx = \sum_{k=0}^{n-1} \int_{a+kT}^{a+kT+T} f(x) dx$$

$$(1) \int_a^{a+T} f(x) dx = \int_0^T f(x) dx$$

将 $a+kT$ 看作 (1) 中的 a , 则有

$$\int_{a+kT}^{a+kT+T} f(x) dx = \int_0^T f(x) dx$$

$$= n \int_0^T f(x) dx \quad (n \in \mathbf{N})$$

$$\begin{aligned} \int_0^{n\pi} \sqrt{1 + \sin 2x} dx \\ = n \int_0^{\pi} \sqrt{1 + \sin 2x} dx \end{aligned}$$

$\sqrt{1 + \sin 2x}$ 是以 π 为周期的周期函数



$$\int_0^{n\pi} \sqrt{1 + \sin 2x} \, dx = n \int_0^{\pi} \sqrt{1 + \sin 2x} \, dx$$

$$= n \int_0^{\pi} \sqrt{(\cos x + \sin x)^2} \, dx$$

$$= n \int_0^{\pi} |\cos x + \sin x| \, dx$$

$$= n\sqrt{2} \int_0^{\pi} \left| \sin\left(x + \frac{\pi}{4}\right) \right| \, dx$$

$$\downarrow \quad \text{令 } t = x + \frac{\pi}{4}$$

$$= n\sqrt{2} \int_{\frac{\pi}{4}}^{\frac{5\pi}{4}} |\sin t| \, dt$$

$$= n\sqrt{2} \int_0^{\pi} |\sin t| \, dt$$

$$= n\sqrt{2} \int_0^{\pi} \sin t \, dt = 2\sqrt{2}n$$

$$\begin{aligned} (1) \int_a^{a+T} f(x) \, dx \\ = \int_0^T f(x) \, dx \end{aligned}$$



二、定积分的分部积分法

定理2. 设 $u(x), v(x) \in C^1[a, b]$, 则

$$\int_a^b u(x) v'(x) dx = u(x)v(x) \Big|_a^b - \int_a^b u'(x) v(x) dx$$

证: $\because [u(x)v(x)]' = u'(x)v(x) + u(x)v'(x)$

↓ 两端在 $[a, b]$ 上积分

$$u(x)v(x) \Big|_a^b = \int_a^b u'(x)v(x) dx + \underline{\int_a^b u(x)v'(x) dx}$$

$$\therefore \int_a^b u(x)v'(x) dx = u(x)v(x) \Big|_a^b - \int_a^b u'(x)v(x) dx$$



例5. 计算 $\int_0^{\frac{1}{2}} \arcsin x \, dx$.

解: 原式 = $x \arcsin x \Big|_0^{\frac{1}{2}} - \int_0^{\frac{1}{2}} \frac{x}{\sqrt{1-x^2}} \, dx$

$$= \frac{\pi}{12} + \frac{1}{2} \int_0^{\frac{1}{2}} (1-x^2)^{-\frac{1}{2}} \, d(1-x^2)$$
$$= \frac{\pi}{12} + (1-x^2)^{\frac{1}{2}} \Big|_0^{\frac{1}{2}}$$
$$= \frac{\pi}{12} + \frac{\sqrt{3}}{2} - 1$$



例6. 证明 $I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$

$$= \begin{cases} \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{3}{4} \cdot \frac{1}{2} \cdot \frac{\pi}{2}, & n \text{ 为偶数} \\ \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{4}{5} \cdot \frac{2}{3}, & n \text{ 为奇数} \end{cases}$$

证: 令 $t = \frac{\pi}{2} - x$, 则

$$\int_0^{\frac{\pi}{2}} \sin^n x \, dx = -\int_{\frac{\pi}{2}}^0 \sin^n \left(\frac{\pi}{2} - t\right) \, dt = \int_0^{\frac{\pi}{2}} \cos^n x \, dx$$

令 $u = \sin^{n-1} x$, $v' = \sin x$,

则 $u' = (n-1)\sin^{n-2} x \cos x$, $v = -\cos x$

$$\therefore I_n = \left[-\cos x \cdot \sin^{n-1} x \right] \Big|_0^{\frac{\pi}{2}} + (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx$$



$$\begin{aligned}
 I_n &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x \cos^2 x \, dx \\
 &= (n-1) \int_0^{\frac{\pi}{2}} \sin^{n-2} x (1 - \sin^2 x) \, dx \\
 &= (n-1) I_{n-2} - (n-1) I_n
 \end{aligned}$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^n x \, dx$$

由此得递推公式 $I_n = \frac{n-1}{n} I_{n-2}$

于是
$$I_{2m} = \frac{2m-1}{2m} \cdot \frac{2m-3}{2m-2} \cdot \dots \cdot \frac{3}{4} \cdot \frac{1}{2} \cdot I_0$$

$$I_{2m+1} = \frac{2m}{2m+1} \cdot \frac{2m-2}{2m-1} \cdot \dots \cdot \frac{4}{5} \cdot \frac{2}{3} \cdot I_1$$

而
$$I_0 = \int_0^{\frac{\pi}{2}} dx = \frac{\pi}{2}, \quad I_1 = \int_0^{\frac{\pi}{2}} \sin x \, dx = 1$$

故所证结论成立.



内容小结

基本积分法 $\left\{ \begin{array}{l} \text{换元积分法} \\ \text{分部积分法} \end{array} \right.$

换元**必**换限
配元**不**换限
边积边代限

思考与练习

1. $\frac{d}{dx} \int_0^x \sin^{100}(x-t) dt = \underline{\sin^{100} x}$

提示: 令 $u = x - t$, 则

$$\int_0^x \sin^{100}(x-t) dt = - \int_x^0 \sin^{100} u du$$



2. 设 $f(t) \in C^1$, $\underline{f(1) = 0, \int_1^{x^3} f'(t) dt = \ln x}$, 求 $f(e)$.

解法1. $\ln x = \int_1^{x^3} f'(t) dt = f(x^3) - f(1) = f(x^3)$

令 $u = x^3$, 得 $f(u) = \ln \sqrt[3]{u} = \frac{1}{3} \ln u \implies f(e) = \frac{1}{3}$

解法2. 对已知等式两边求导,

得 $3x^2 f'(x^3) = \frac{1}{x}$

令 $u = x^3$, 得 $f'(u) = \frac{1}{3u}$

$$\begin{aligned}\therefore f(e) &= \int_1^e f'(u) du + f(1) \\ &= \frac{1}{3} \int_1^e \frac{1}{u} du = \frac{1}{3}\end{aligned}$$

思考: 若改题为

$$\int_1^{x^3} f'(\sqrt[3]{t}) dt = \ln x$$

$f(e) = ?$

提示: 两边求导, 得

$$f'(x) = \frac{1}{3x^3}$$

$$f(e) = \int_1^e f'(x) dx$$



3. 设 $f''(x)$ 在 $[0,1]$ 连续, 且 $f(0)=1, f(2)=3, f'(2)=5$,
求 $\int_0^1 x f''(2x) dx$.

解: $\int_0^1 x \underline{f''(2x)} dx = \frac{1}{2} \int_0^1 x df'(2x)$ (分部积分)

$$= \frac{1}{2} \left[x f'(2x) \Big|_0^1 - \int_0^1 f'(2x) dx \right]$$

$$= \frac{5}{2} - \frac{1}{4} f(2x) \Big|_0^1$$

$$= 2$$



5.3 作业

P254-255

1 (4) , (10) , (16) ,(24) ;

3 ; 7 (4), (9), (10)



备用题

1. 证明 $f(x) = \int_x^{x+\frac{\pi}{2}} |\sin x| \, dx$ 是以 π 为周期的函数.

$$\begin{aligned} \text{证: } f(x+\pi) &= \int_{x+\pi}^{x+\pi+\frac{\pi}{2}} |\sin u| \, du \\ &\quad \downarrow \text{令 } u = t + \pi \\ &= \int_x^{x+\frac{\pi}{2}} |\sin(t+\pi)| \, dt \\ &= \int_x^{x+\frac{\pi}{2}} |\sin t| \, dt = \int_x^{x+\frac{\pi}{2}} |\sin x| \, dx \\ &= f(x) \end{aligned}$$

$\therefore f(x)$ 是以 π 为周期的周期函数.



2. 设 $f(x)$ 在 $[a, b]$ 上有连续的二阶导数, 且 $f(a) = f(b) = 0$, 试证 $\int_a^b f(x) dx = \frac{1}{2} \int_a^b (x-a)(x-b) \underline{f''(x)} dx$

证: 右端 $= \frac{1}{2} \int_a^b (x-a)(x-b) df'(x)$ 分部积分

$$= \frac{1}{2} \left[(x-a)(x-b) f'(x) \right] \Big|_a^b \\ - \frac{1}{2} \int_a^b \underline{f'(x)} (2x-a-b) \underline{dx}$$

$$= -\frac{1}{2} \int_a^b (2x-a-b) df(x)$$
 再次分部积分

$$= -\frac{1}{2} \left[(2x-a-b) f(x) \right] \Big|_a^b + \int_a^b f(x) dx = \text{左端}$$

